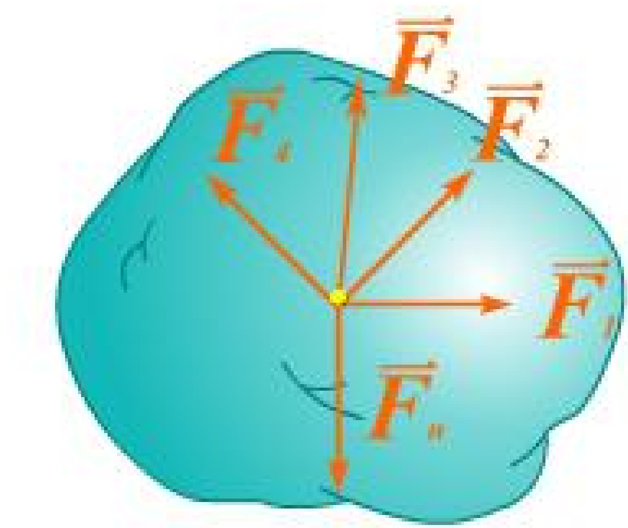
第二章　平面力系

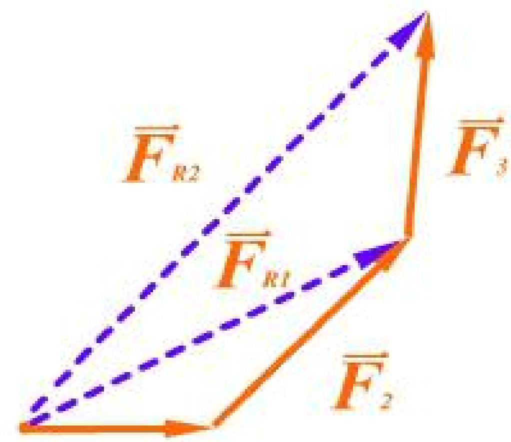
§2-1　平面汇交力系

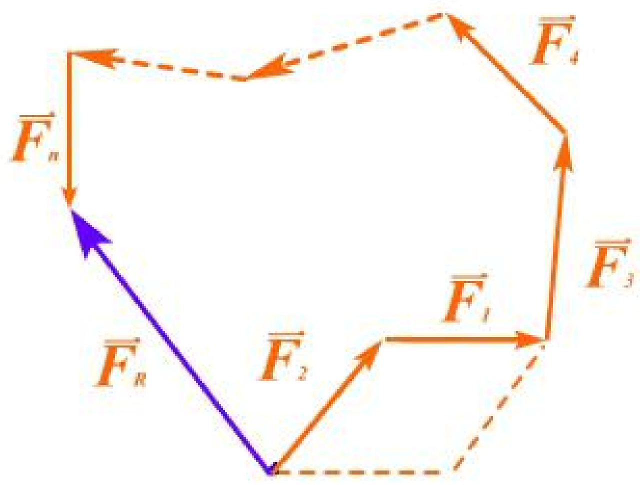
一、平面汇交力系合成的几何法——力多边形规则



$$\vec{F}_{\mathrm{R}1} = \vec{F}_1 + \vec{F}_2$$

$$\vec{F}_{\mathrm{R}2} = \vec{F}_{\mathrm{R}1} + \vec{F}_{\mathrm{R}3} = \sum_{i=1}^{3} \vec{F}_i$$





$$\vec{F}_{\mathrm{R}} = \sum_{i=1}^{n} \vec{F}_i = \sum \vec{F}_i$$

力多边形

力多边形规则

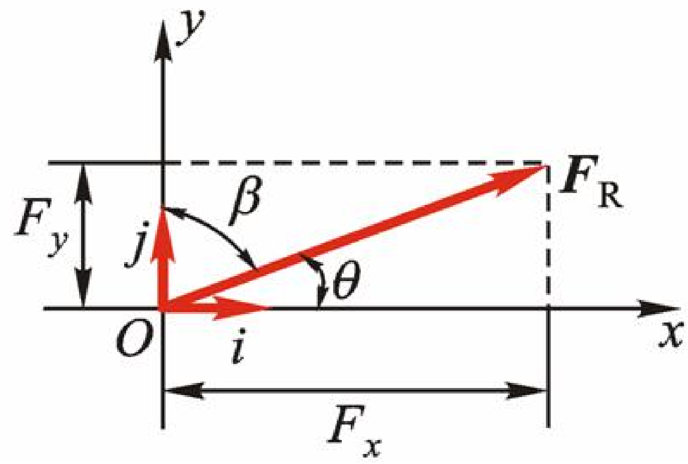
二、平面汇交力系平衡的几何条件

平衡条件 $\sum \vec{F}_i = 0$

平面汇交力系平衡的必要和充分条件是：

该力系的力多边形自行封闭.

三、平面汇交力系合成的解析法



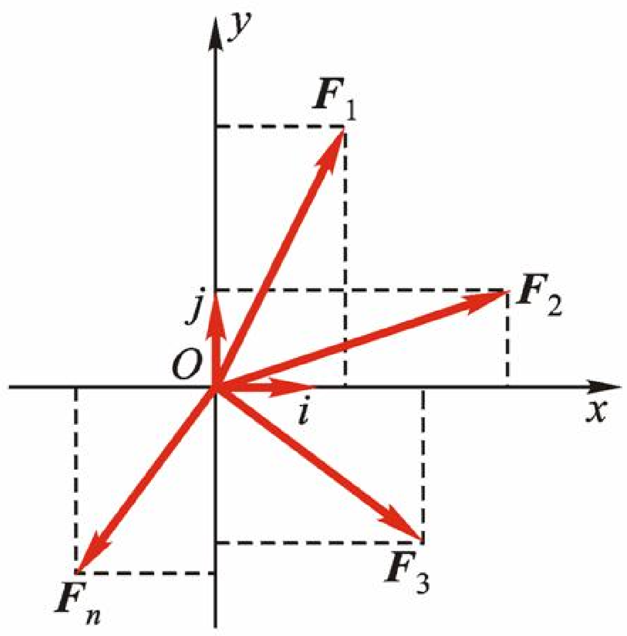
合力 $\vec{F}_R$ 在x轴，y轴投影分别为

$$F_{Rx} = F_R \cos\theta$$

$$F_{Ry} = F_R \cos\beta$$

合力等于各力矢量和

$$\vec{F}_R = \sum \vec{F}_i$$



由合矢量投影定理，得合力投影定理

$$F_{Rx} = \sum F_{ix} \qquad F_{Ry} = \sum F_{iy}$$

合力的大小为：

$$F_{\mathrm{R}} = \sqrt{F_{\mathrm{R}x}{}^2 + F_{\mathrm{R}y}{}^2}$$

方向为：

$$\cos(\vec{F}_{\mathrm{R}}, \vec{i}\,) = \frac{\sum F_{ix}}{F_{\mathrm{R}}} \qquad \cos(\vec{F}_{\mathrm{R}}, \vec{j}\,) = \frac{\sum F_{iy}}{F_{\mathrm{R}}}$$

作用点为力的汇交点.

四、平面汇交力系的平衡方程

平衡条件 $\vec{F}_{\mathrm{R}} = 0$

平衡方程

$$\sum F_x = 0 \qquad \sum F_y = 0$$

例2-1

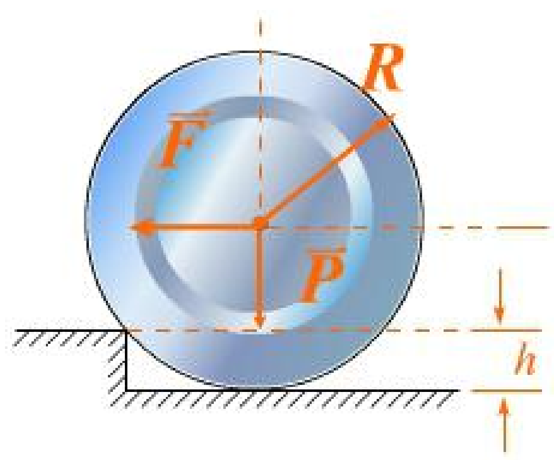
已知： $P = 20\,\text{kN}, R = 0.6\,\text{m}, h = 0.08\,\text{m}$

求：

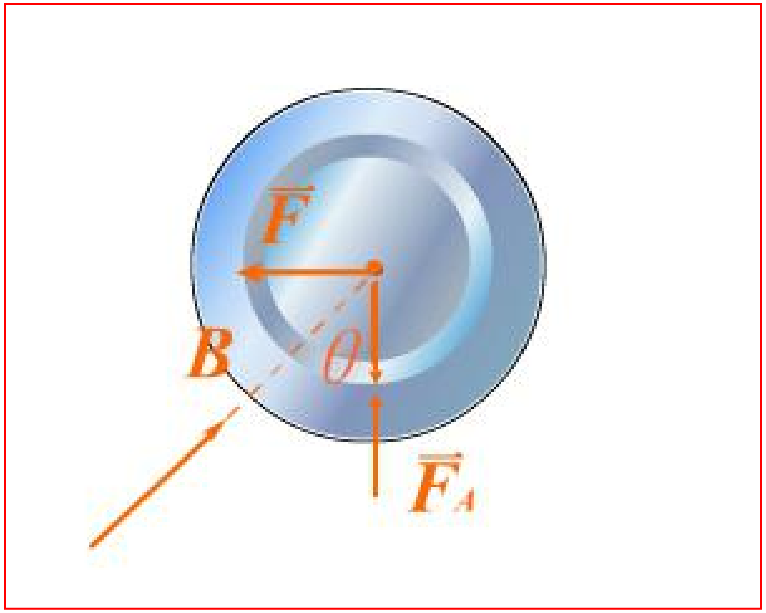
1. 水平拉力 $F = 5\,\text{kN}$ 时，碾子对地面及障碍物的压力？

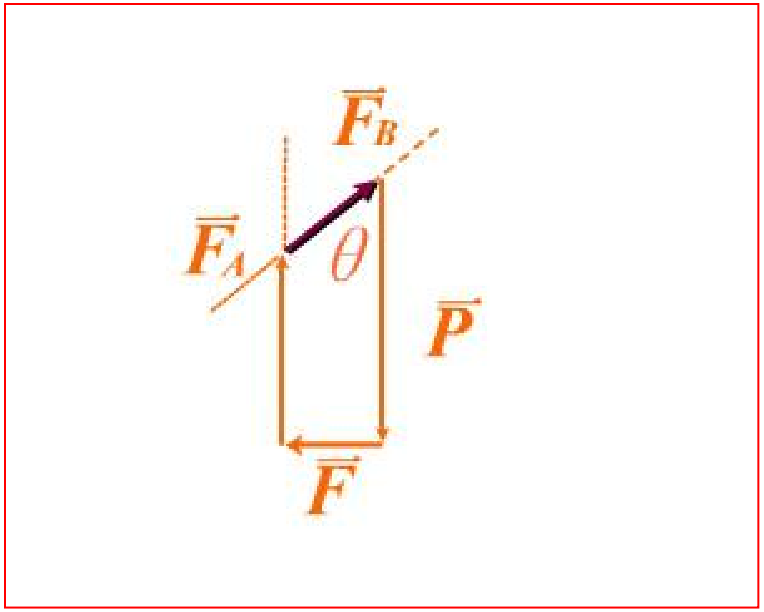
2. 欲将碾子拉过障碍物，水平拉力 $\vec{F}$ 至少多大？

3. 力 $\vec{F}$ 沿什么方向拉动碾子最省力，及此时力 $\vec{F}$ 多大？



解: 1. 取碾子，画受力图.　用几何法，按比例画封闭力四边形





$$\theta = \arccos\frac{R-h}{R} = 30^\circ$$

$$F_B \sin\theta = F$$

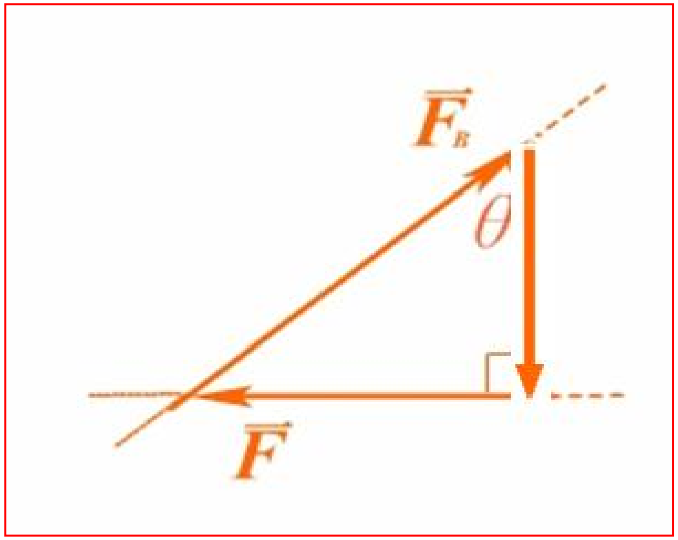
$$F_A + F_B\cos\theta = P$$

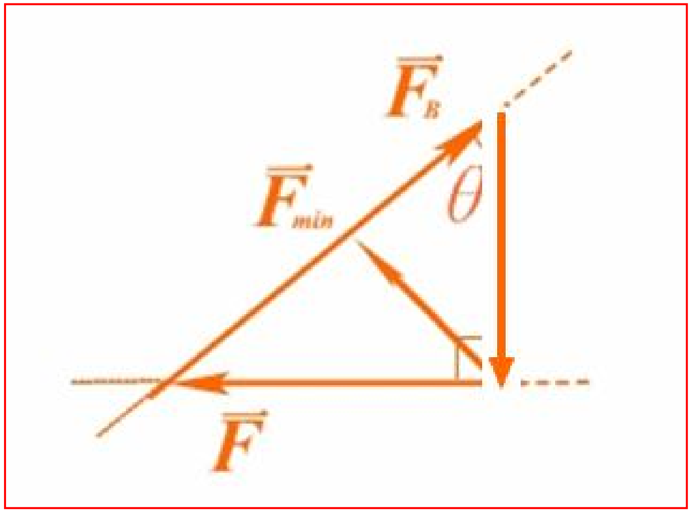
$$F_A = 11.4\text{kN}$$

$$F_B = 10\text{kN}$$

2. 碾子拉过障碍物， 应有 $F_A = 0$

用几何法解得 $F = P \cdot \tan\theta = 11.55\text{kN}$



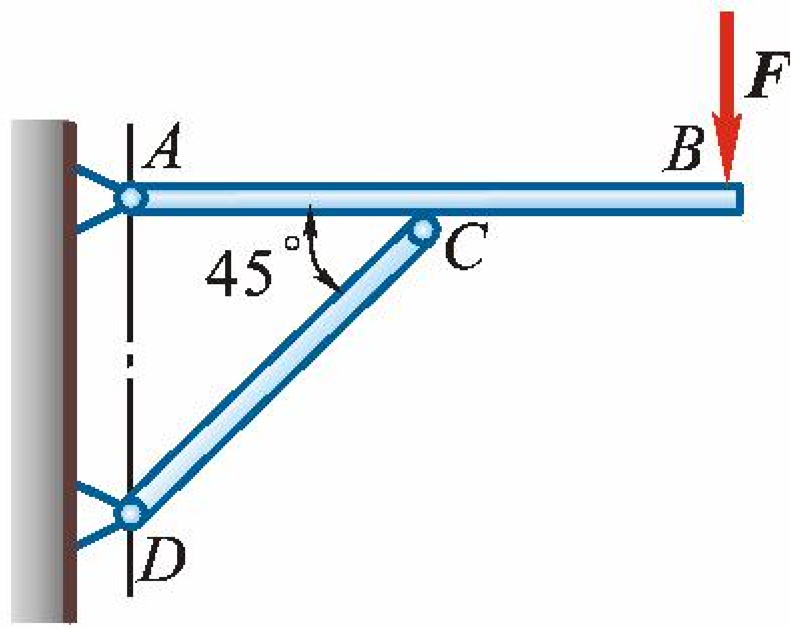


3. 解得 $F_{\min} = P \cdot \sin\theta = 10\ \text{kN}$

例2-2

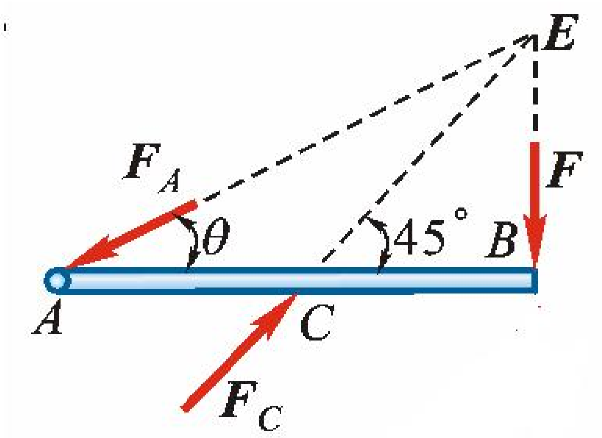
已知：$AC=CB, F=10\,\text{kN}$，各杆自重不计；

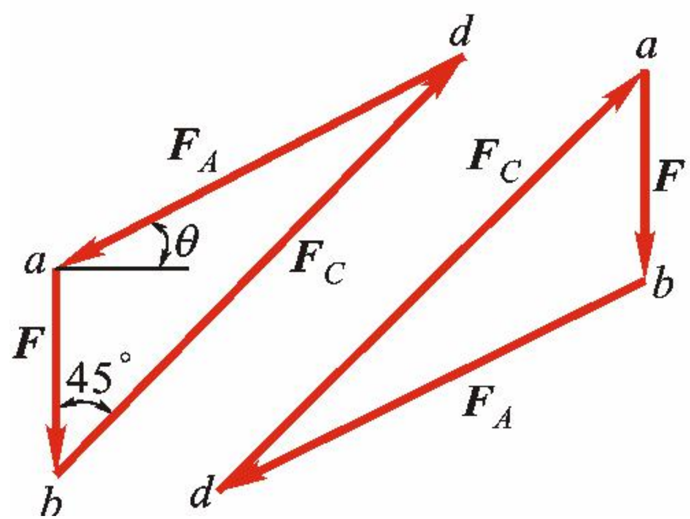
求：CD杆及铰链A的受力.



解： CD为二力杆，取AB杆，画受力图.

用几何法，画封闭力三角形.





按比例量得 $F_C = 28.3\text{kN}, F_A = 22.4\text{kN}$

例2-3 已知：图示平面共点力系，$F_1=200\text{N}$，$F_2=300\text{N}$，$F_3=100\text{N}$，$F_4=250\text{N}$ 求：此力系的合力.

解： 用解析法

$$F_{Rx}=\sum F_{ix}=F_1\cos30^\circ-F_2\cos60^\circ-F_3\cos45^\circ+F_4\cos45^\circ=129.3\text{N}$$

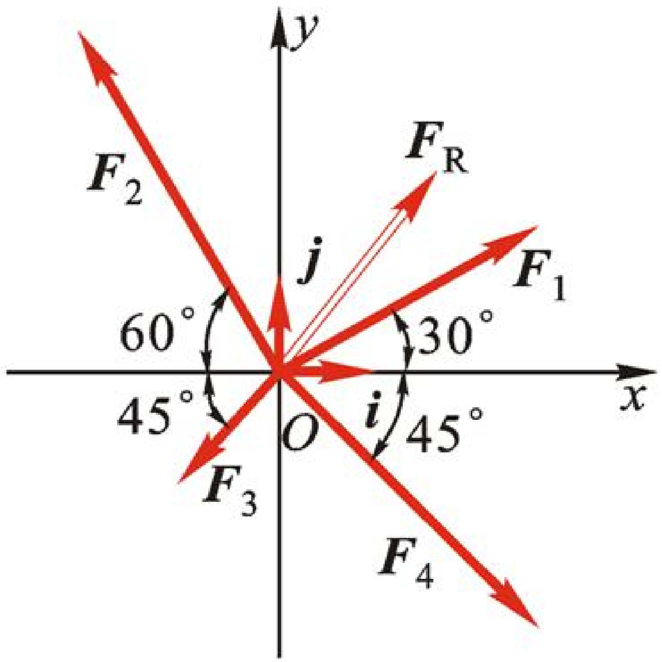
$$F_{Ry}=\sum F_{iy}=F_1\sin30^\circ+F_2\sin60^\circ-F_3\sin45^\circ-F_4\sin45^\circ=112.3\text{N}$$

$$F_R=\sqrt{F_{Rx}^2+F_{Ry}^2}=171.3\text{N}$$

$$\cos\theta=\frac{F_{Rx}}{F_R}=0.7548$$

$$\cos\beta=\frac{F_{Ry}}{F_R}=0.6556$$

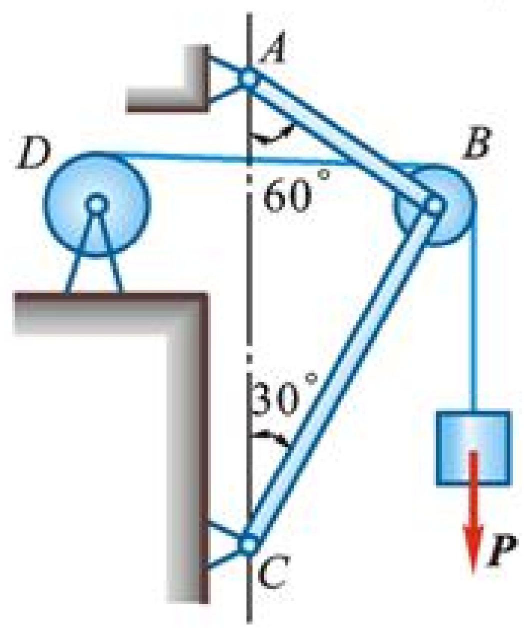
$$\theta=40.99^\circ,\ \beta=49.01^\circ$$



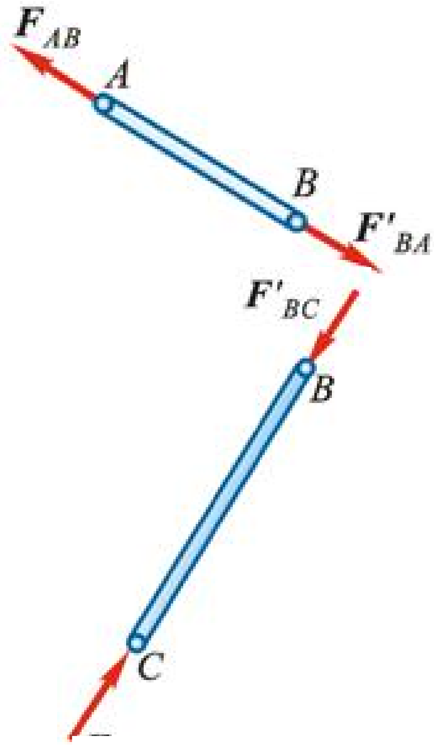
例2-4

已知：系统如图，不计杆、轮自重，忽略滑轮大小，P=20kN；

求：系统平衡时，杆AB，BC受力.



解： ***AB*、*BC*杆为二力杆，取滑轮*B*（或点*B*），画受力图．建图示坐标系**

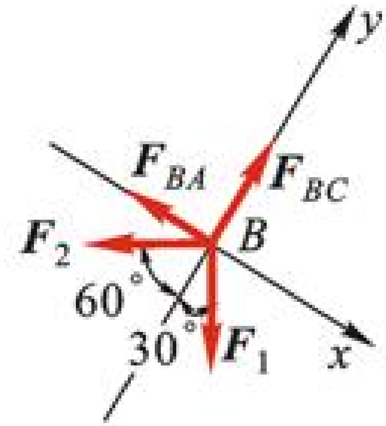


$$\sum F_x = 0 \qquad -F_{BA} + F_1 \cos 60^\circ - F_2 \cos 30^\circ = 0$$

$$\sum F_y = 0 \qquad F_{BC} - F_1 \cos 30^\circ - F_2 \cos 60^\circ = 0$$

$$F_1 = F_2 = P$$

$$F_{BA} = -7.321\text{kN} \qquad F_{BC} = 27.32\text{kN}$$



例2-5

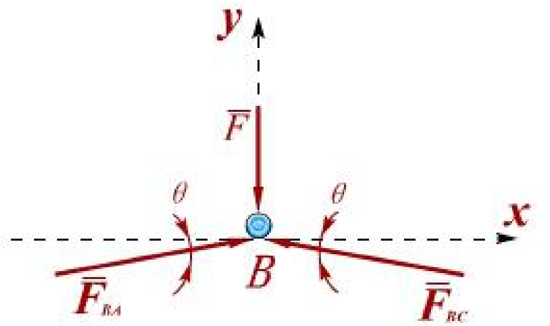
已知： *F*=3kN，*l*=1500mm，*h*=200mm，忽略自重；

求：平衡时，压块*C*对工件与地面的压力，*AB*杆受力.

解：

AB、*BC*杆为二力杆.

取销钉*B*.



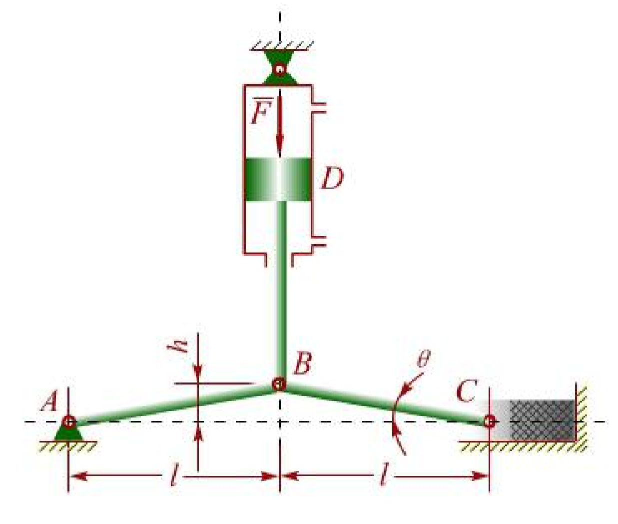
$$\sum F_x = 0 \quad F_{BA}\cos\theta - F_{BC}\cos\theta = 0$$

$$\Rightarrow \quad F_{BA} = F_{BC}$$

$$\sum F_y = 0 \quad F_{BA}\sin\theta + F_{BC}\sin\theta - F = 0$$

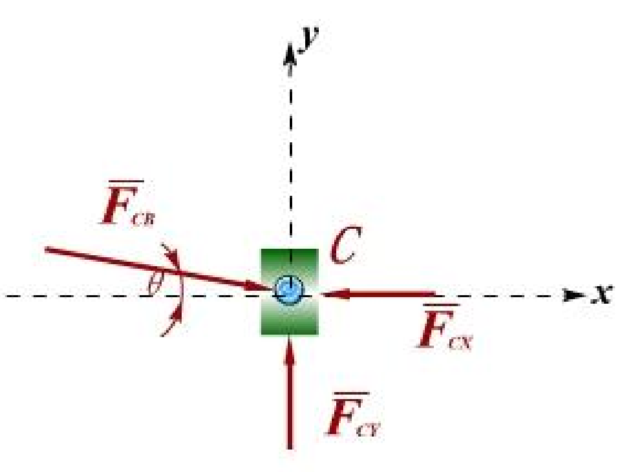
$$\Rightarrow \quad F_{BA} = F_{BC} = 11.35\text{kN}$$

选压块C



$$\sum F_x = 0 \qquad F_{CB}\cos\theta - F_{Cx} = 0$$

$$F_{Cx} = \frac{F}{2}\cot\theta = \frac{Fl}{2h} = 11.25\text{kN}$$

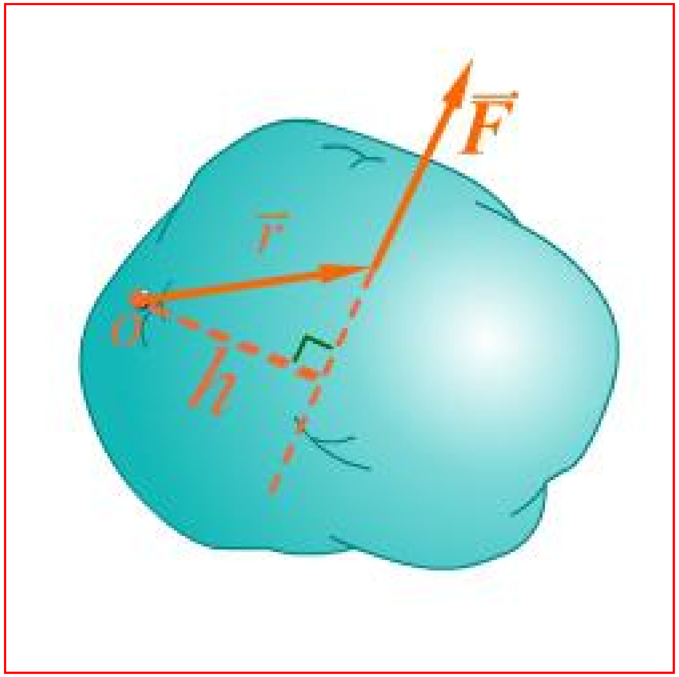


$$\sum F_y = 0 \qquad -F_{CB}\sin\theta + F_{Cy} = 0$$

$$F_{Cy} = 1.5\text{kN}$$

§2-2 平面力对点之矩 平面力偶理论

一、平面力对点之矩（力矩）



力矩作用面，O 称为矩心，O 到力的作用线的垂直距离 h 称为力臂

两个要素：

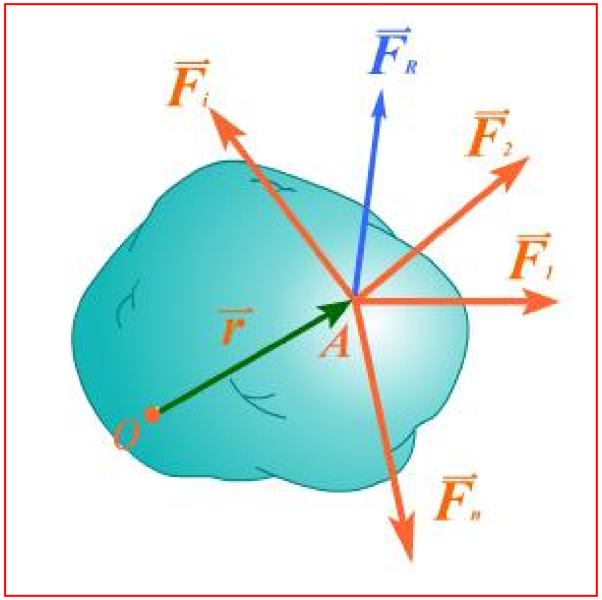
1. 大小：力 $\vec{F}$ 与力臂的乘积
2. 方向：转动方向

$$M_O(\vec{F}) = \pm F \cdot h$$

力对点之矩是一个代数量，它的绝对值等于力的大小与力臂的乘积，它的正负：力使物体绕矩心逆时针转向时为正，反之为负．常用单位 $N \cdot m$ 或 $kN \cdot m$

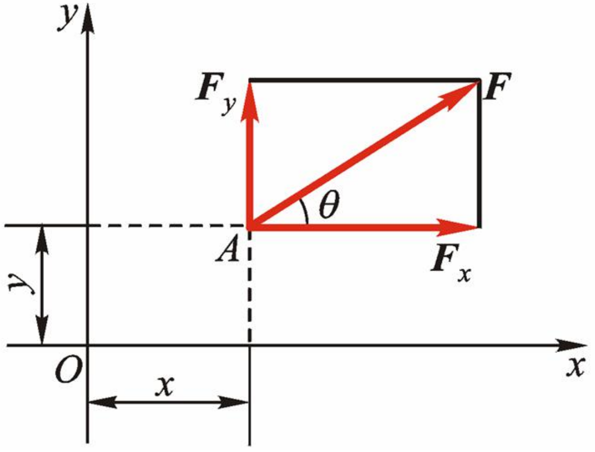
二、合力矩定理与力矩的解析表达式

合力矩定理：平面汇交力系的合力对平面内任一点之矩等于所有各分力对于该点之矩的代数和。



$$M_O(\vec{F}_R) = \sum M_O(\vec{F}_i)$$

该结论适用于任何合力存在的力系



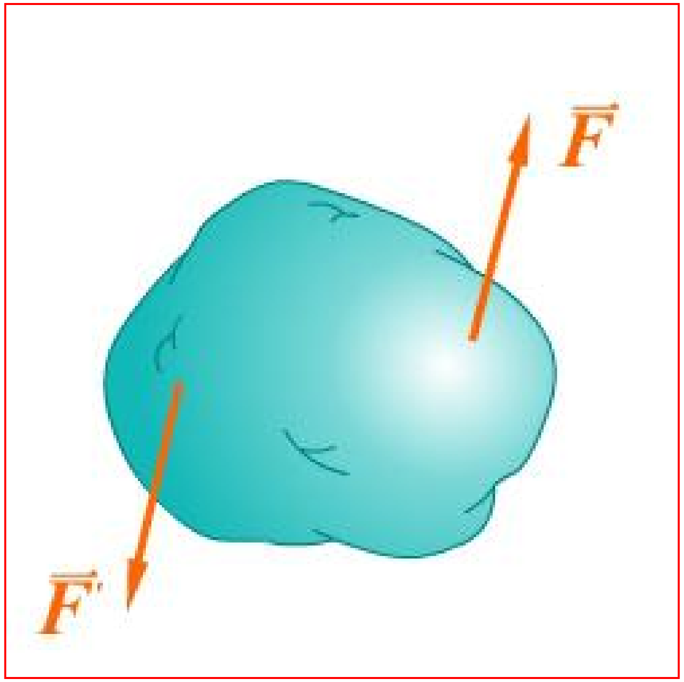
$$M_O(\vec{F}) = M_O(\vec{F}_y) - M_O(\vec{F}_x)$$
$$= x \cdot F \cdot \sin\theta - y \cdot F \cdot \cos\theta$$
$$= xF_y - yF_x$$

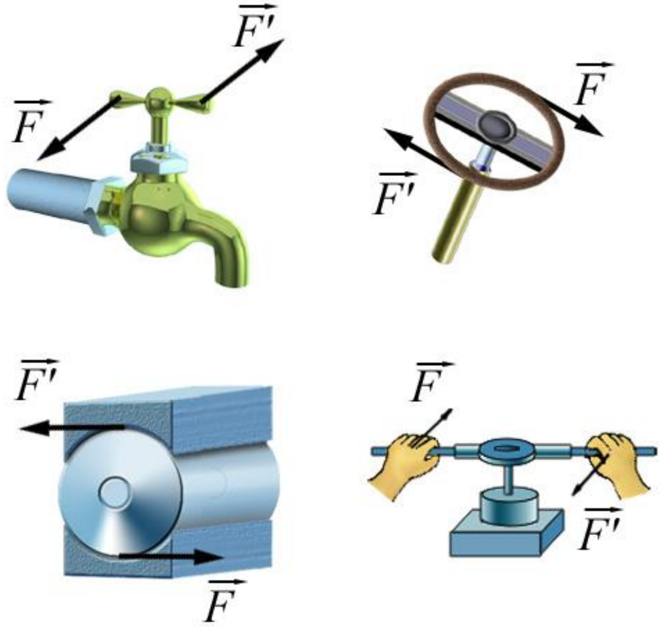
$$M_O(\vec{F}_R) = \sum (x_i \cdot F_{iy} - y_i \cdot F_{ix})$$

三、力偶和力偶矩

力偶

由两个等值、反向、不共线的（平行）力组成的力系称为力偶，记作 $(\vec{F},\vec{F}')$





力偶矩

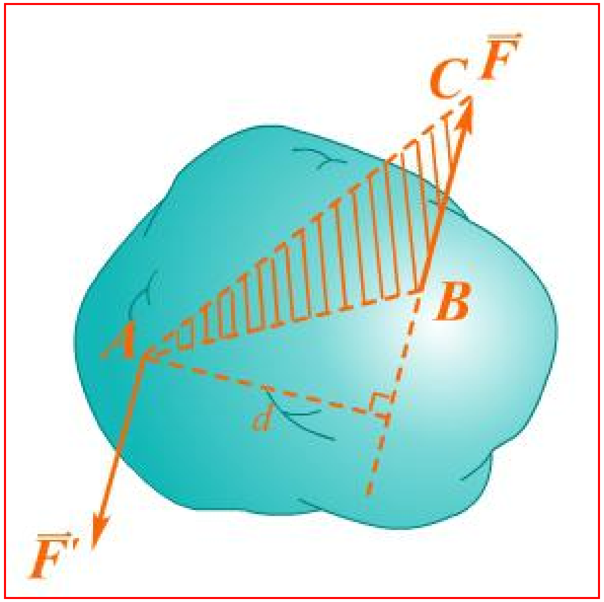
力偶中两力所在平面称为力偶作用面.

力偶两力之间的垂直距离称为力偶臂.

两个要素

a. 大小：力与力偶臂乘积

b. 方向：转动方向



力偶矩

$$M = \pm F \cdot d = \pm 2\Delta ABC$$

四、同平面内力偶的等效定理

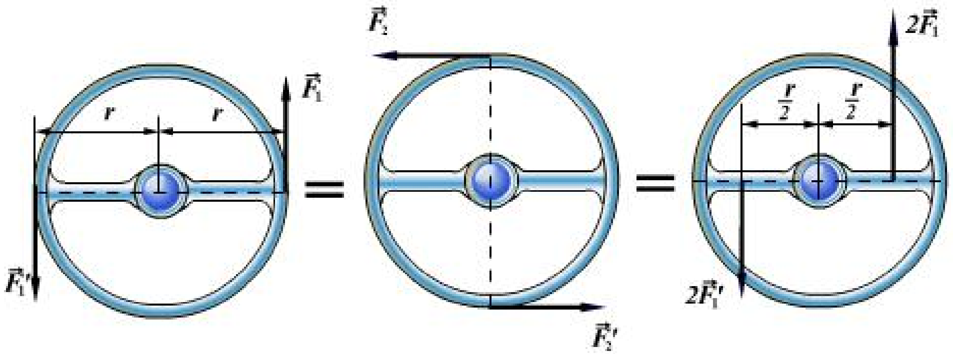
定理： 同平面内的两个力偶，如果力偶矩相等，则两力偶彼此等效。

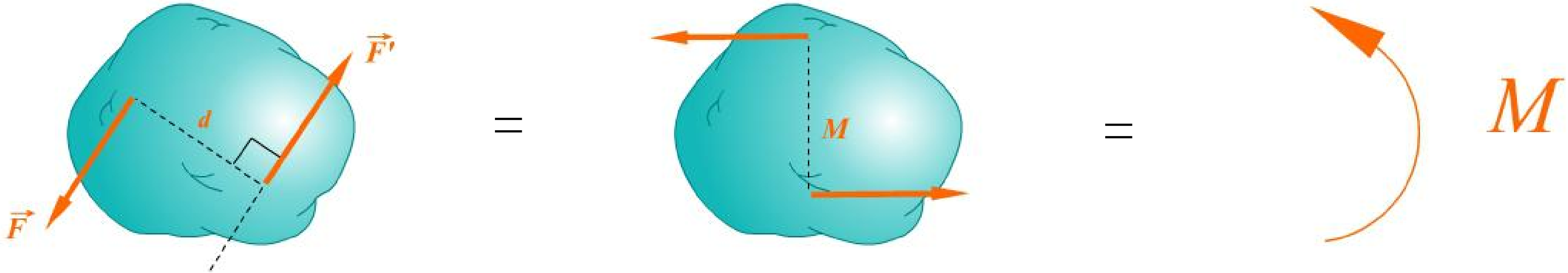
推论：

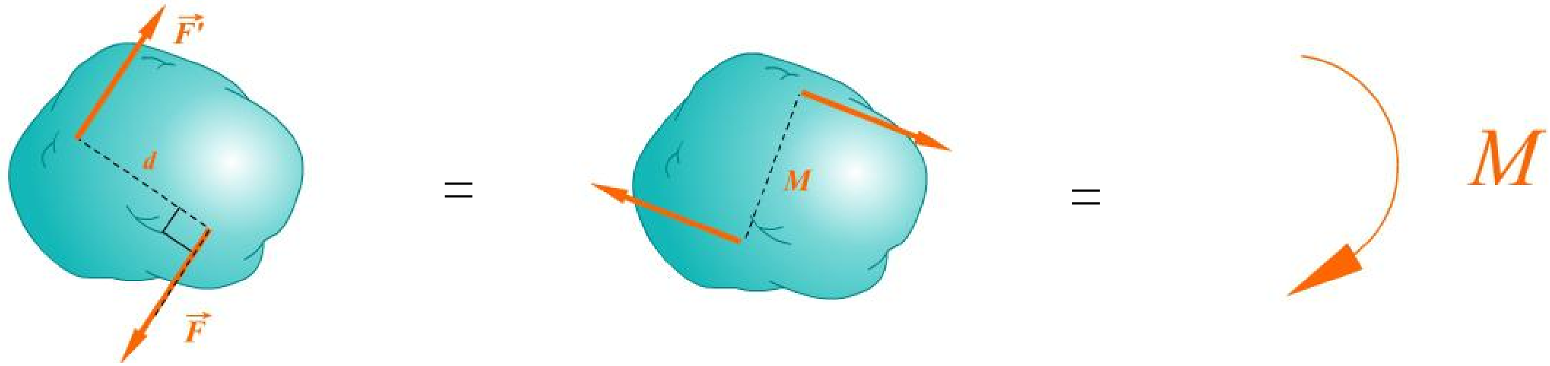
任一力偶可在它的作用面内任意转移，而不改变它对刚体的作用。因此力偶对刚体的作用与力偶在其作用面内的位置无关。

只要保持力偶矩不变，可以同时改变力偶中力的大小与力偶臂的长短，对刚体的作用效果不变.

力偶中的力偶臂和力的大小都不是力偶的特征量，只有力偶矩是平面力偶作用的唯一度量。


$\vec{F}_1$
r
r
$\vec{F}_1'$
$\vec{F}_2$
$\vec{F}_2'$
$2\vec{F}_1$
$\frac{r}{2}$
$\frac{r}{2}$
$2\vec{F}_1'$


$\vec{F}'$
d
$\vec{F}$
=
M
=
M


$\vec{F}'$
d
$\vec{F}$
=
M
=
M

五、平面力偶系的合成和平衡条件

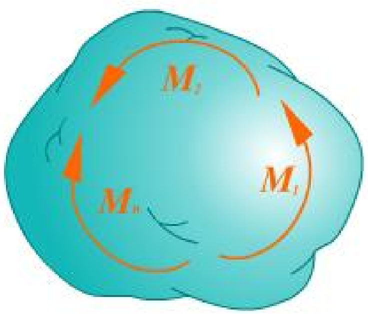
已知： $M_1, M_2, \cdots M_n$；

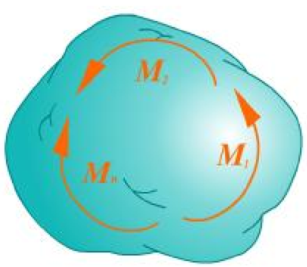
任选一段距离 d

$$\frac{M_1}{d} = F_1 \qquad M_1 = F_1 d$$

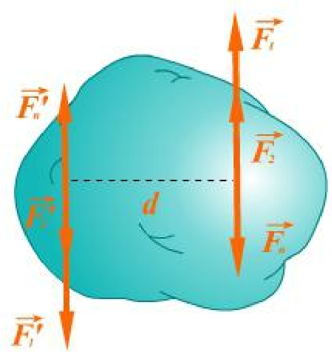
$$\frac{M_2}{d} = F_2 \qquad M_2 = F_2 d$$

$$\left|\frac{M_n}{d}\right| = F_n \qquad M_n = -F_n d$$

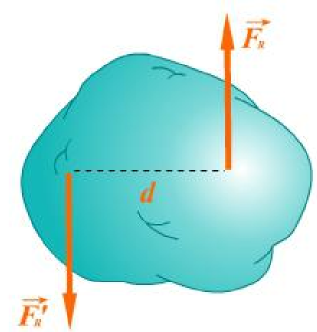




=

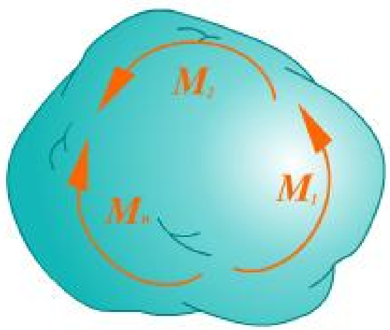


=

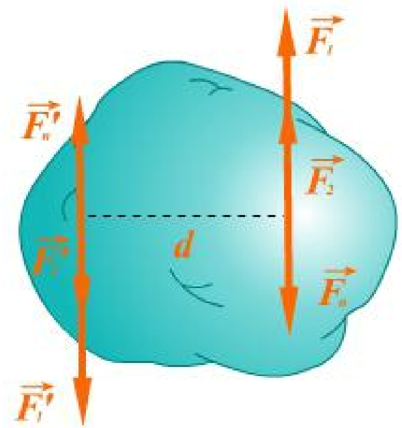


$$F_{\mathrm{R}} = F_1 + F_2 + \cdots - F_n$$

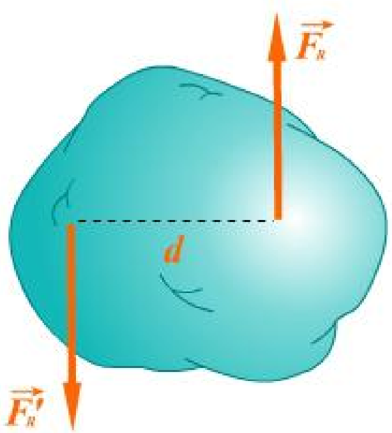
$$F'_{\mathrm{R}} = F'_1 + F'_2 + \cdots - F'_n$$



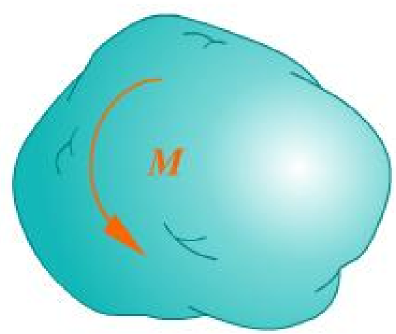
=



=



=



$$M = F_R d = F_1 d + F_2 d + \cdots - F_n d = M_1 + M_2 + \cdots M_n$$

$$M = \sum_{i=1}^{n} M_i = \sum M_i$$

平面力偶系平衡的充要条件 $M = 0$ ，有如下平衡方程

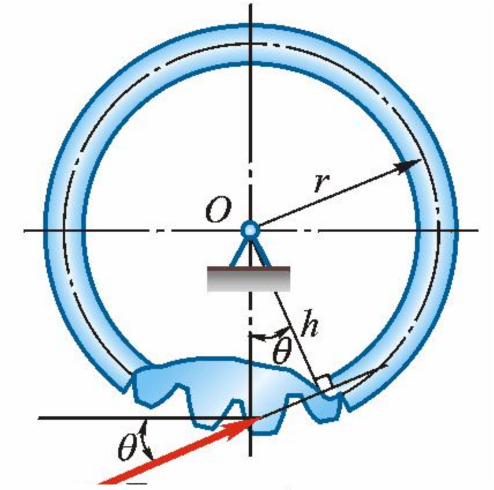
$$\sum M_i = 0$$

平面力偶系平衡的必要和充分条件是：所有各力偶矩的代数和等于零.

例2-6

已知：$F=1400\text{N}$， $\theta=20^\circ$， $r=60\text{mm}$

求：$M_O(\vec{F})$

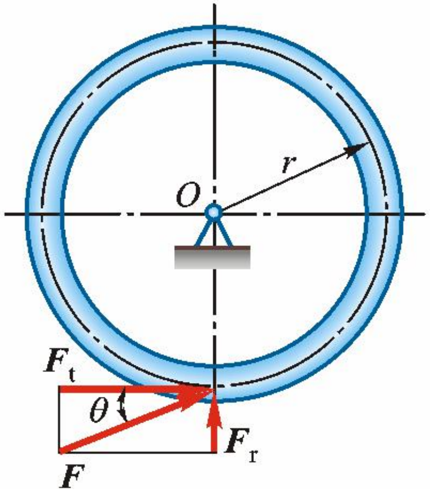


解: 直接按定义

$$M_O\left(\vec{F}\right)=F\cdot h=F\cdot r\cdot\cos\theta$$

$$=78.93\text{N}\cdot\text{m}$$

按合力矩定理



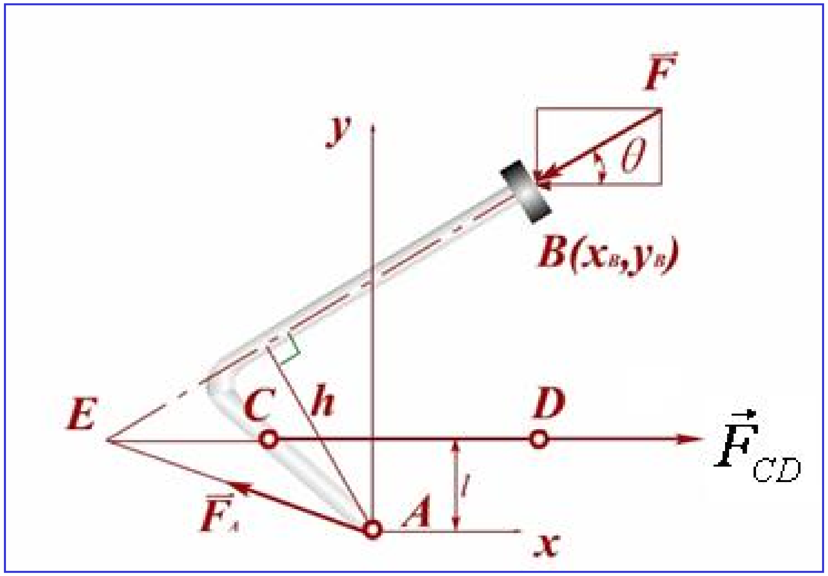
$$M_O\left(\vec{F}\right)=M_O\left(\vec{F}_t\right)+M_O\left(\vec{F}_r\right)$$

$$=F\cdot\cos\theta\cdot r=78.93\text{N}\cdot\text{m}$$

例2-7

已知：$F, \theta, x_B, y_B, l;$

求： 平衡时，CD 杆的拉力.



解： CD 为二力杆，取踏板

由杠杆平衡条件

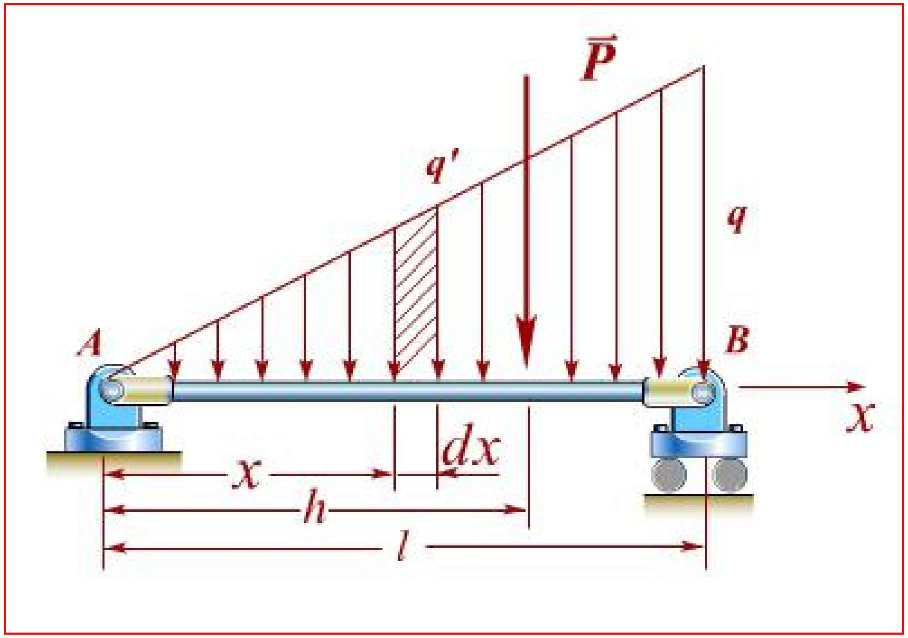
$$F\cos\theta \cdot y_B - F\sin\theta \cdot x_B - F_{CD} \cdot l = 0$$

解得 $$F_{CD} = \frac{F\cos\theta \cdot y_B - F\sin\theta \cdot x_B}{l}$$

例2-8 已知：q,l;

求：合力及合力作用线位置.

解： 取微元如图



$$q' = \frac{x}{l} \cdot q$$

$$P = \int_0^l \frac{x}{l} \cdot q \cdot \mathrm{d}x = \frac{1}{2} ql$$

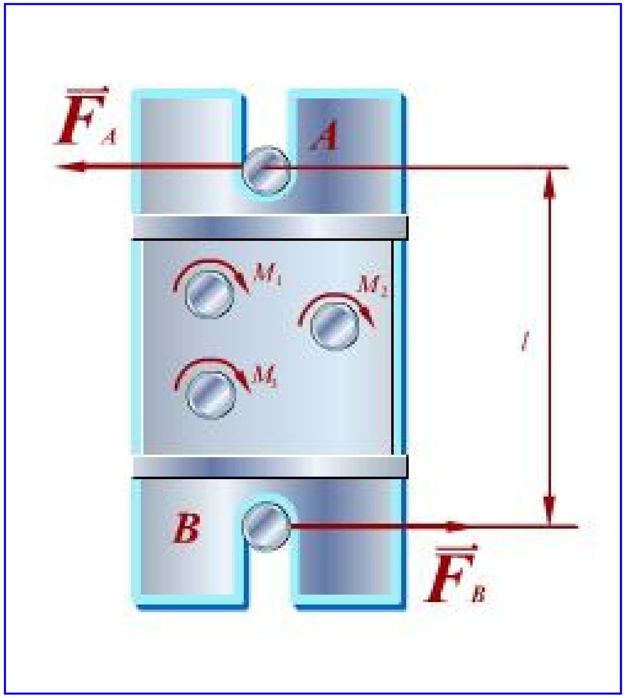
由合力矩定理 $P \cdot h = \int_0^l q' \cdot \mathrm{d}x \cdot x = \int_0^l \frac{x^2}{l} q \cdot \mathrm{d}x$

得 $h = \dfrac{2}{3} l$

例2-9

已知：$M_1 = M_2 = 10\ \mathbf{N \cdot m}, M_3 = 20\ \mathbf{N \cdot m}, l = 200\ \mathbf{mm}$；

求：　光滑螺柱 AB 所受水平力.



解： 由力偶只能由力偶平衡的性质，其受力图为

$$\sum M = 0$$

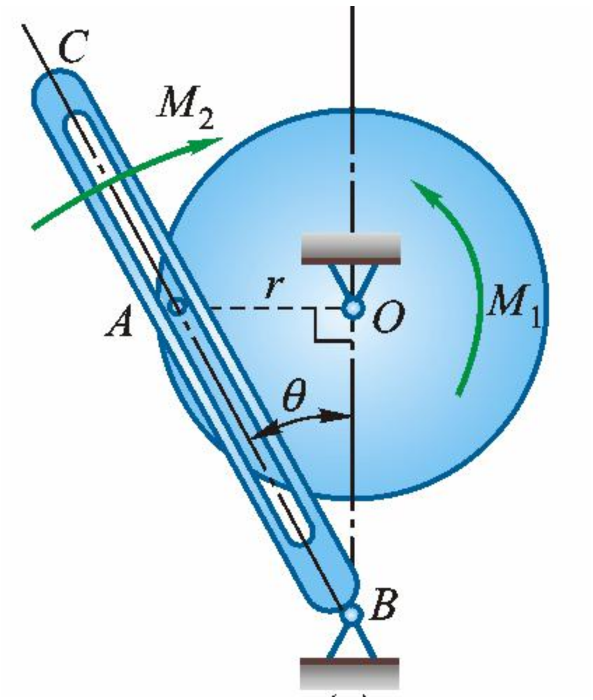
$$F_A l - M_1 - M_2 - M_3 = 0$$

解得 $$F_A = F_B = \frac{M_1 + M_2 + M_3}{l} = 200\ \mathbf{N}$$

例2-10

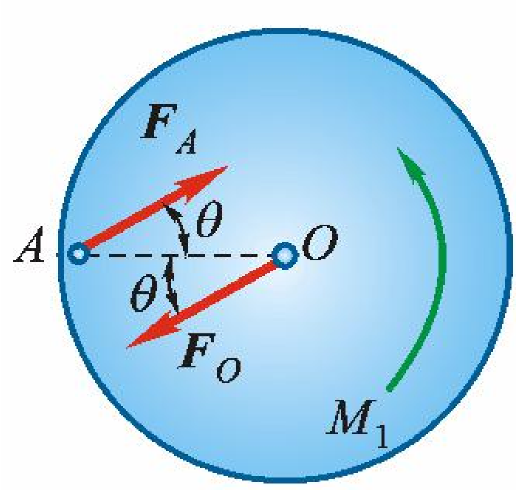
已知 $M_1 = 2\text{kN}\cdot\text{m}, OA = r = 0.5\text{m}, \theta = 30^\circ;$

求：平衡时的 M_2 及铰链 O, B 处的约束力.



解： 取轮，由力偶只能由力偶平衡的性质，画受力图.

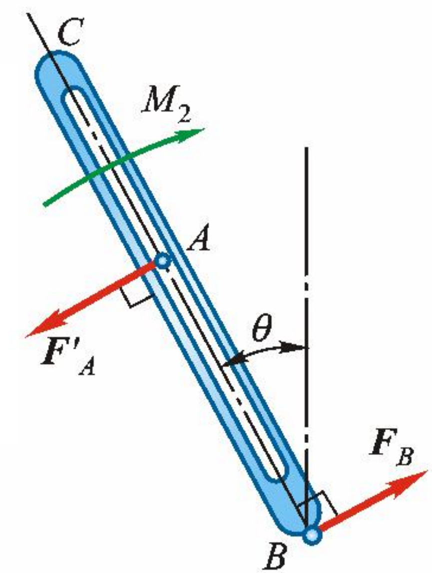
$$\sum M = 0 \qquad M_1 - F_A \cdot r\sin\theta = 0$$



解得 $\qquad F_O = F_A = 8\text{kN}$

取杆 BC，画受力图.

$$\sum M = 0 \qquad F_A' \cdot \frac{r}{\sin\theta} - M_2 = 0$$



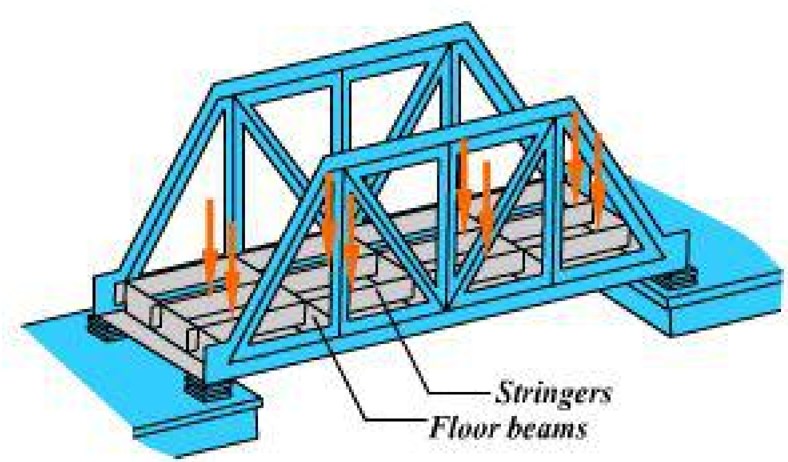
解得 $\qquad M_2 = 8\text{kN}\cdot\text{m}$

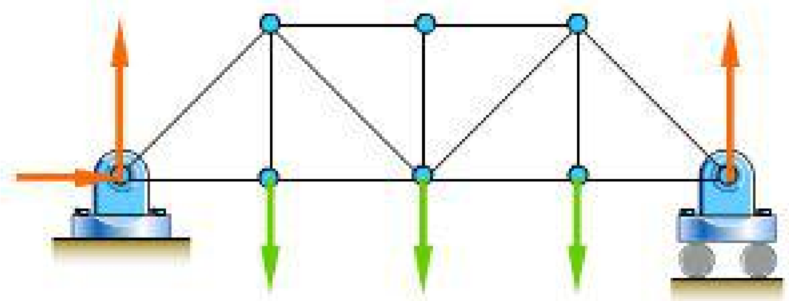
$$F_B = F_A = 8\text{kN}$$

§2-3　平面任意力系的简化

当力系中各力的作用线处于同一平面内且任意分布时，称其为平面任意力系.

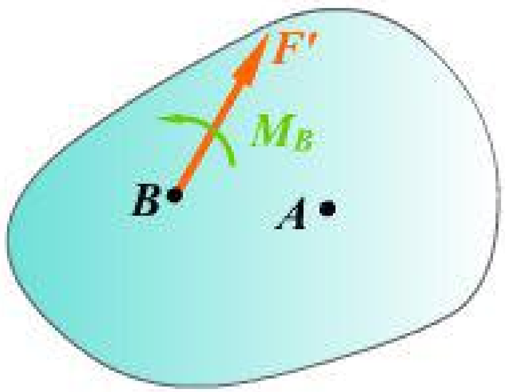
平面任意力系实例



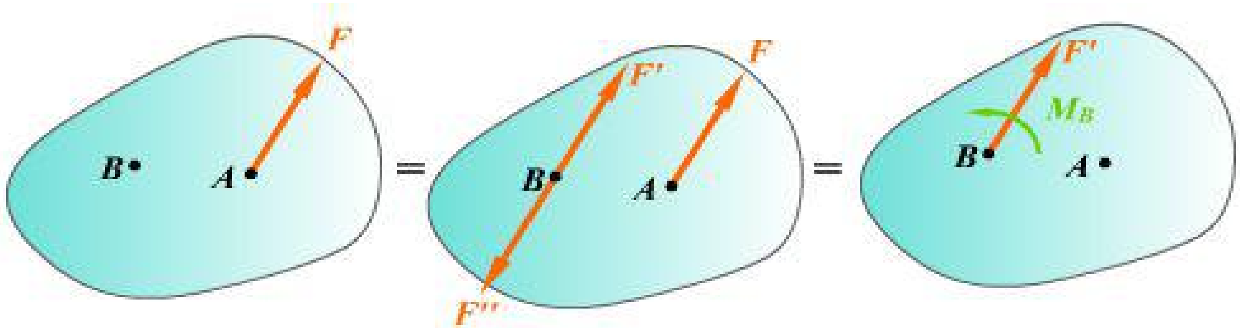


一. 力的平移定理

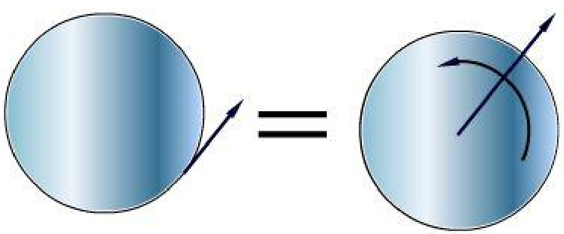
可以把作用在刚体上点 A 的力 F 平行移到任一点 B，但必须同时附加一个力偶，这个附加力偶的矩等于原来的力 F 对新作用点 B 的矩.



$$M_B = M_B(\vec{F}) = Fd$$



实例

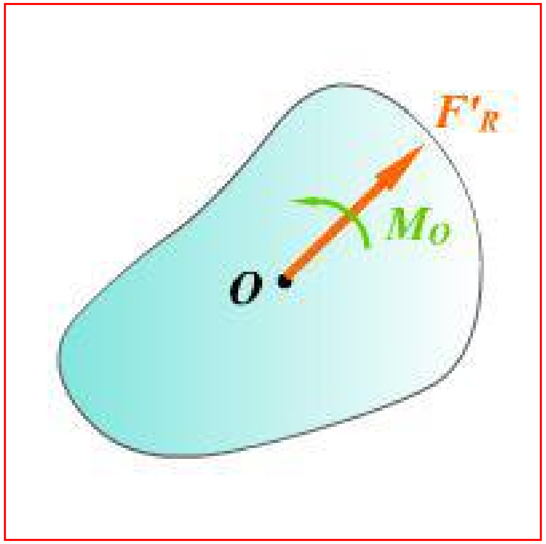


二. 平面任意力系向作用面内一点简化 • 主矢和主矩

$$\vec{F}_1' = \vec{F}_1 \quad M_1 = M_O(\vec{F}_1)$$

$$\vec{F}_2' = \vec{F}_2 \quad M_2 = M_O(\vec{F}_2)$$

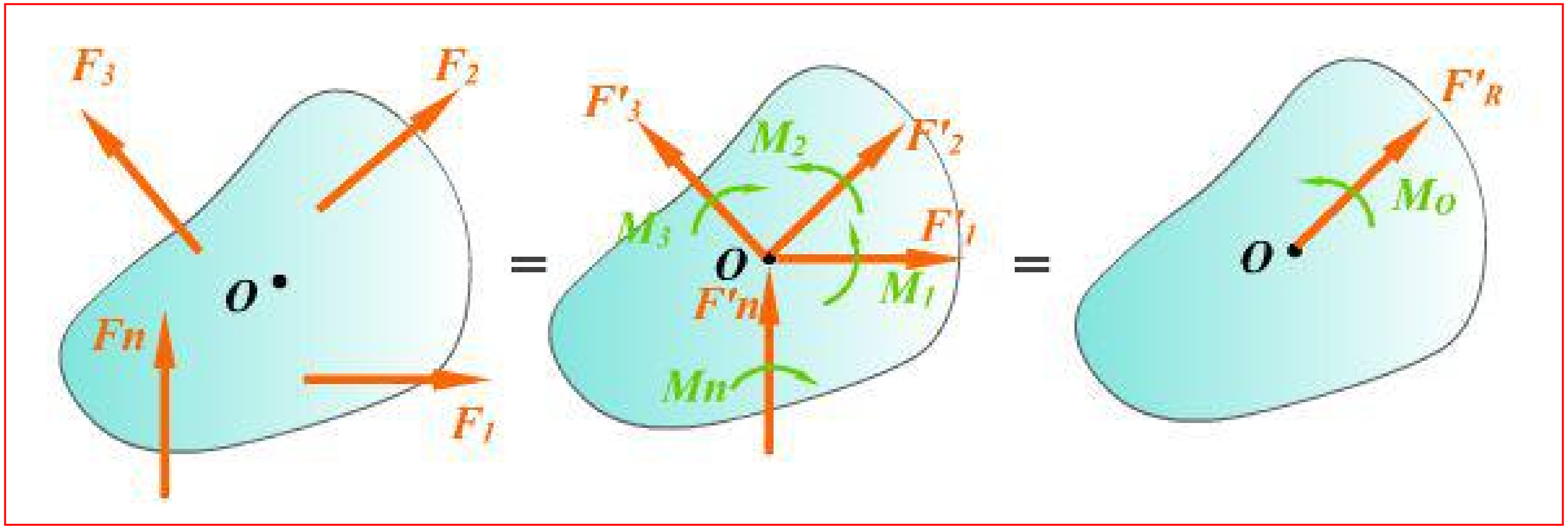
$$\vec{F}_n' = \vec{F}_n \quad M_n = M_O(\vec{F}_n)$$



$$\vec{F}_{\mathrm{R}}' = \sum \vec{F}_i' = \sum \vec{F}_i$$

$$\boldsymbol{M}_O = \sum \boldsymbol{M}_i = \sum \boldsymbol{M}_O(\vec{\boldsymbol{F}}_i)$$

主矢 $\vec{F}_{\mathrm{R}}' = \sum \vec{F}_i$ 主矩 $M_O = \sum M_O(\vec{F}_i)$



主矢与简化中心无关，而主矩一般与简化中心有关.

$$F_{Rx}' = \sum F_{ix}' = \sum F_{ix} = \sum F_x$$

$$F_{Ry}' = \sum F_{iy}' = \sum F_{iy} = \sum F_y$$

主矢大小

$$F_R' = \sqrt{(\sum F_{ix})^2 + (\sum F_{iy})^2}$$

方向

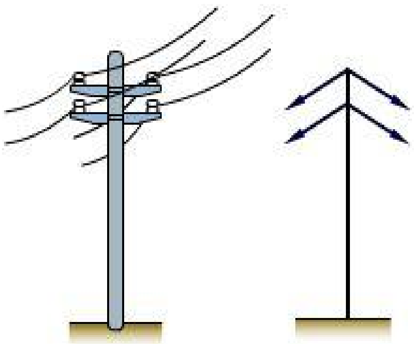
$$\cos(\vec{F}'_R, \vec{i}) = \frac{\sum F_{ix}}{F_R'} \qquad \cos(\vec{F}'_R, \vec{j}) = \frac{\sum F_{iy}}{F_R'}$$

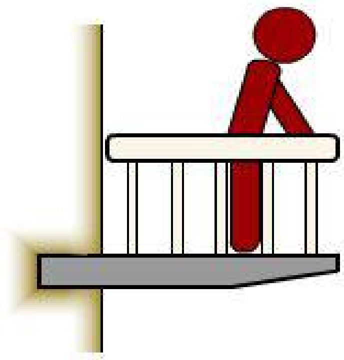
作用点　　作用于简化中心上

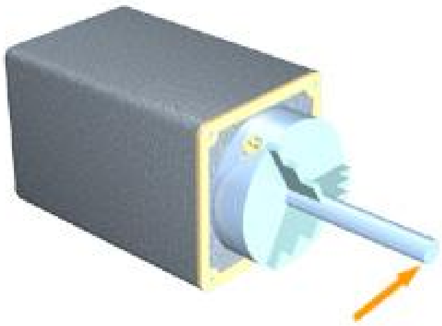
主矩

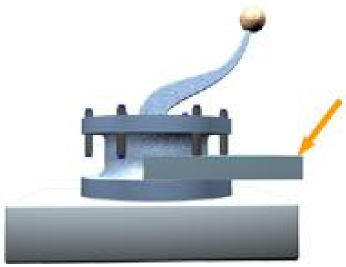
$$\boldsymbol{M_O} = \sum \boldsymbol{M_O}(\vec{\boldsymbol{F}}_i)$$

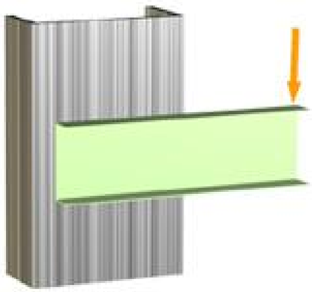
平面固定端约束

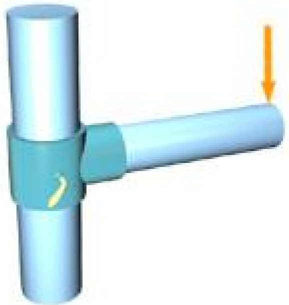


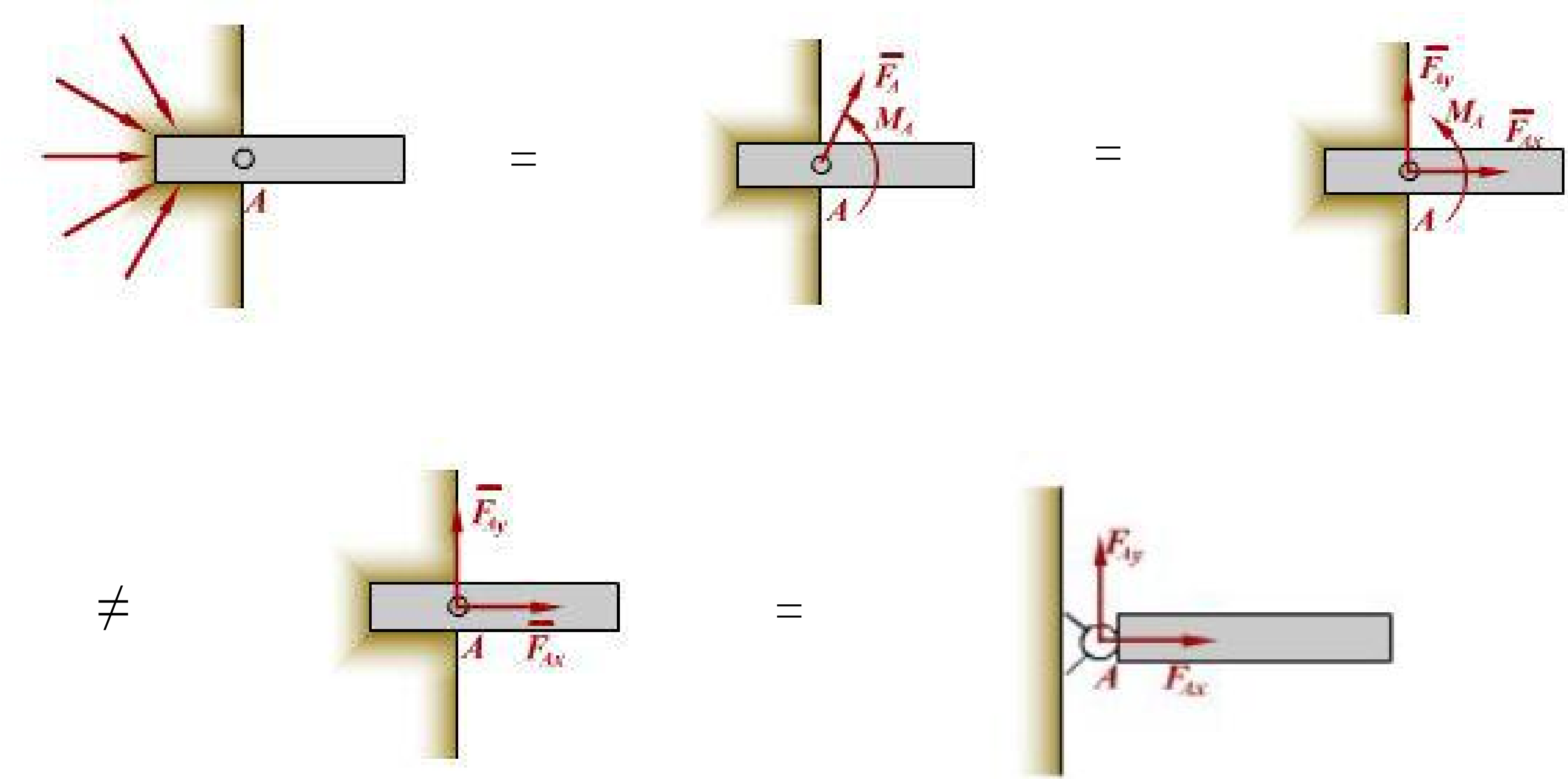








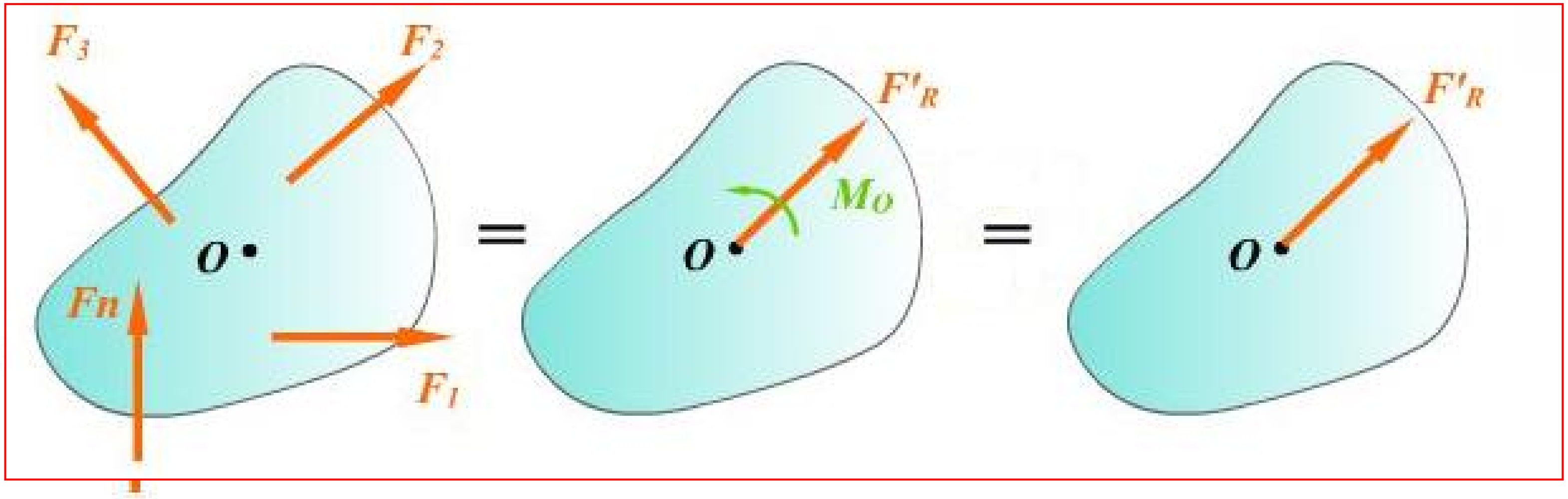




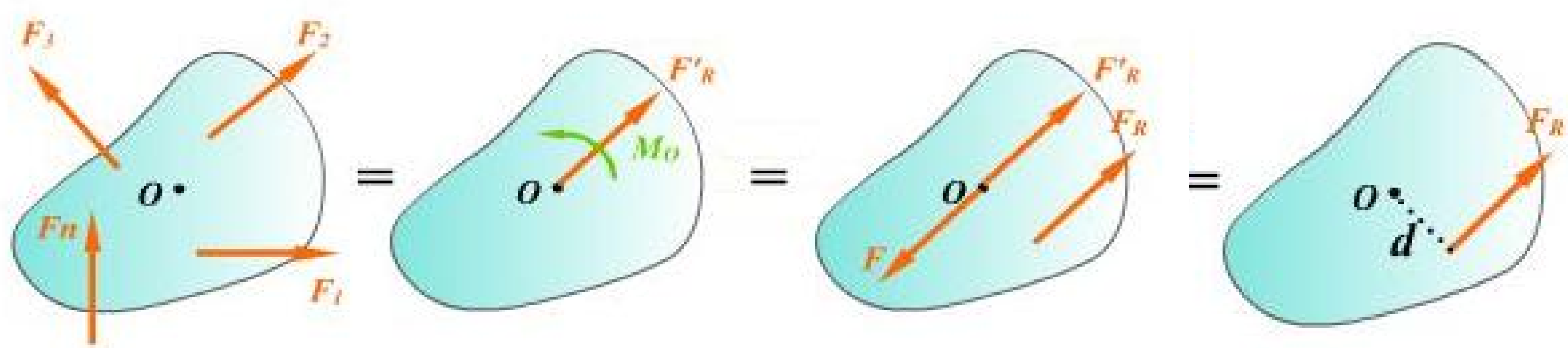
A
=
$\overline{F}_A$
M_A
A
=
$\overline{F}_{Ay}$
M_A
$\overline{F}_{Ax}$
A
≠
$\overline{F}_{Ay}$
A
$\overline{F}_{Ax}$
=
F_{Ay}
A
F_{Ax}

三. 平面任意力系的简化结果分析

$\overline{F}_{\mathrm{R}}' \neq 0 \quad M_O = 0$ ⟶ 合力作用线过简化中心



$\overline{F}'_{\mathrm{R}} \neq 0 \quad M_O \neq 0$ ⟹ 合力，作用线距简化中心 $M_O / |F'_{\mathrm{R}}|$

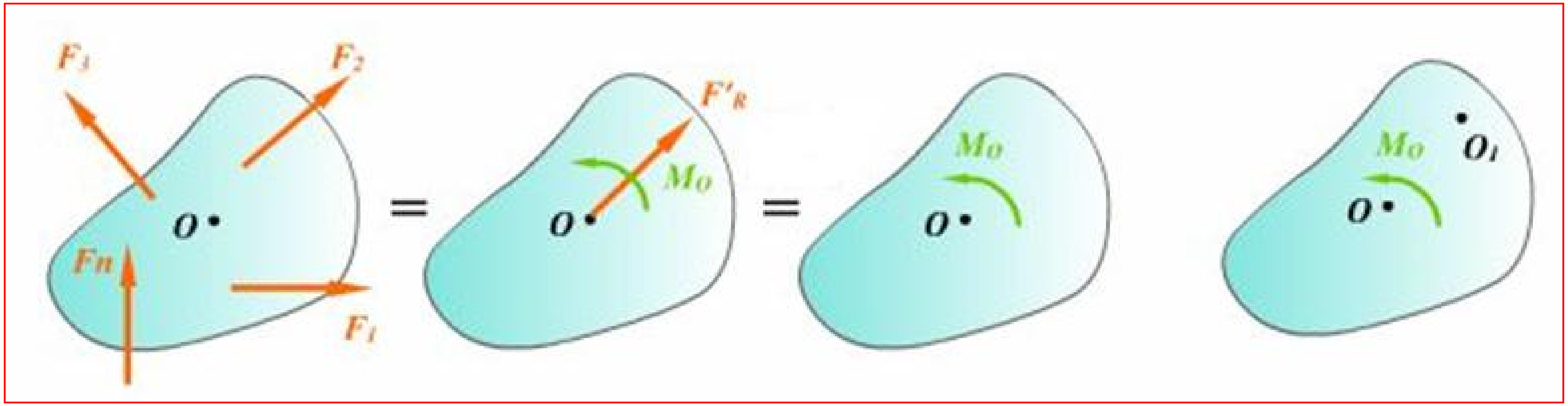


$$d = \frac{M_O}{F'_{\mathrm{R}}} \qquad M_O = F'_{\mathrm{R}} d \qquad F_{\mathrm{R}} = F'_{\mathrm{R}} = F$$

合力矩定理 $M_O(\overline{F}_{\mathrm{R}}) = M_O = \sum M_O(\overline{F}_i)$

$\overline{F}'_{\mathrm{R}} = 0 \qquad M_O \neq 0$ → 合力偶

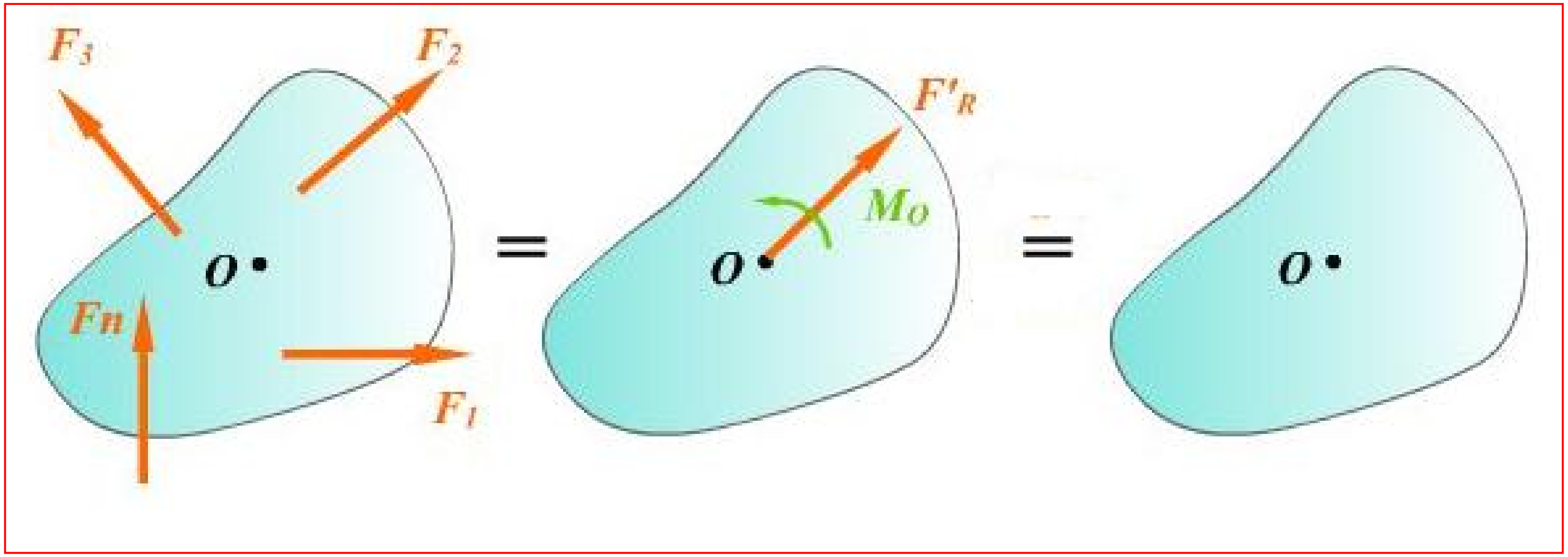
与简化中心的位置无关



若为 O_1 点，如何?

$\overline{F}_{\mathrm{R}}' = 0 \quad M_O = 0 \quad \Longrightarrow$ 平衡

与简化中心的位置无关



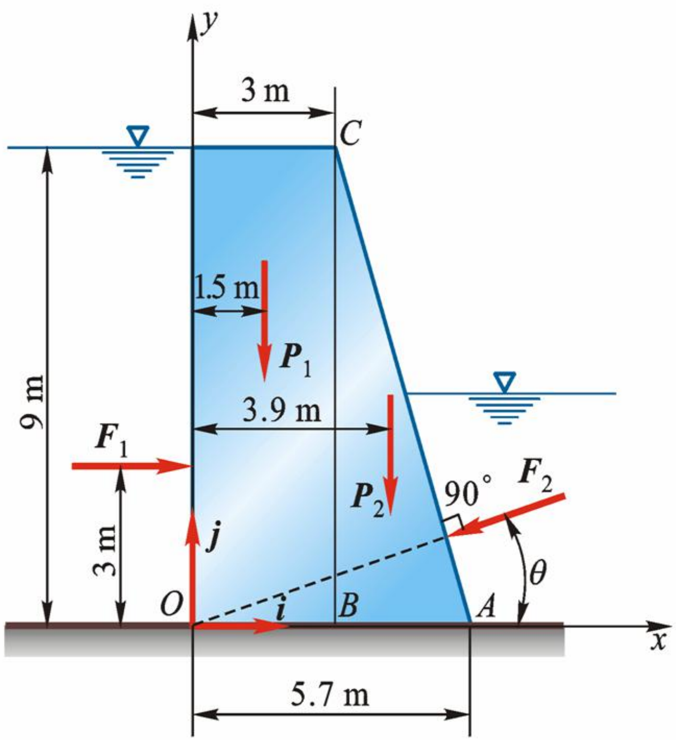
例2-11

已知：$P_1 = 450\text{kN}, P_2 = 200\text{kN}, F_1 = 300\text{kN}, F_1 = 70\text{kN}$

求：力系向 O 点的简化结果；

合力与OA 的交点到点O 的距离 x ；

合力作用线方程。

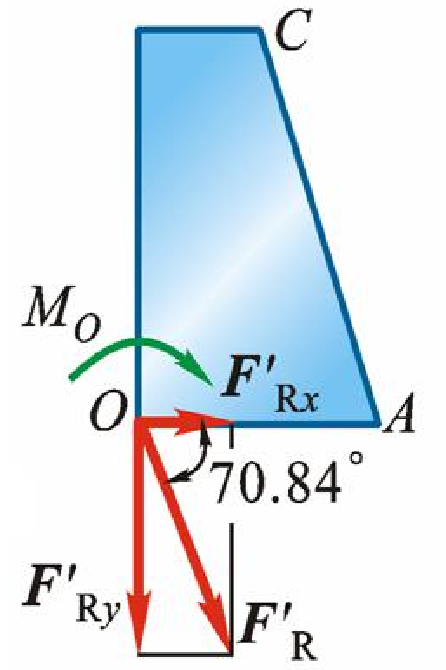


解：

（1）主矢：

$$\sum F_x = F_1 - F_2\cos\theta = 232.9\text{kN}$$

$$\sum F_y = -P_1 - P_2 - F_2\sin\theta = -670.1\text{kN}$$



$$F_R' = \sqrt{(\sum F_x)^2 + (\sum F_y)^2} = 709.4\text{kN}$$

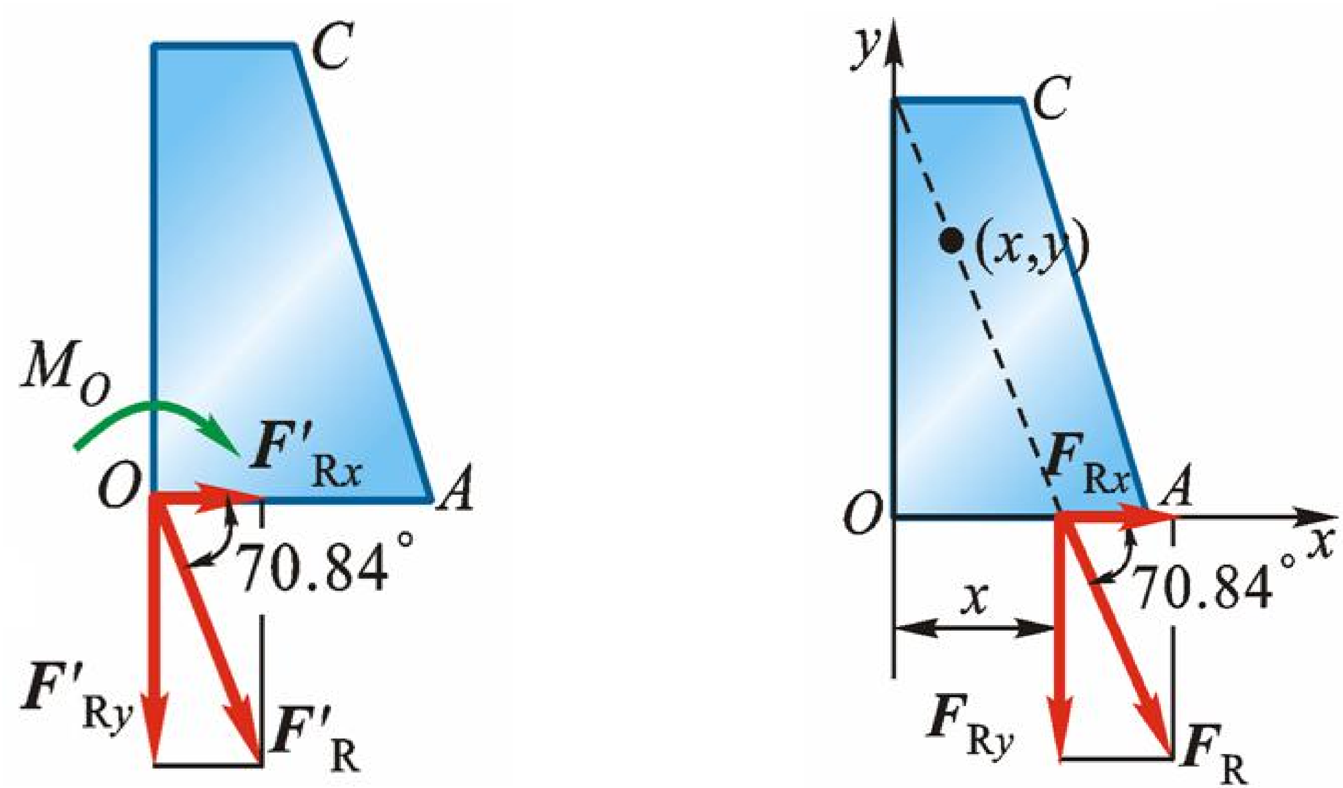
$$\cos(\vec{F}_R', \vec{i}) = \frac{\sum F_x}{F_R'} = 0.3283, \quad \cos(\vec{F}_R', \vec{j}) = \frac{\sum F_y}{F_R'} = -0.9446$$

$$\angle(\vec{F}_R', \vec{i}) = \pm 70.84^\circ, \quad \angle(\vec{F}_R', \vec{j}) = 180^\circ \pm 19.16^\circ$$

主矩：

$$M_O = \sum M_O(\vec{F}) = -3F_1 - 1.5P_1 - 3.9P_2 = -2355\text{kN}\cdot\text{m}$$

（2）求合力及其作用线位置：



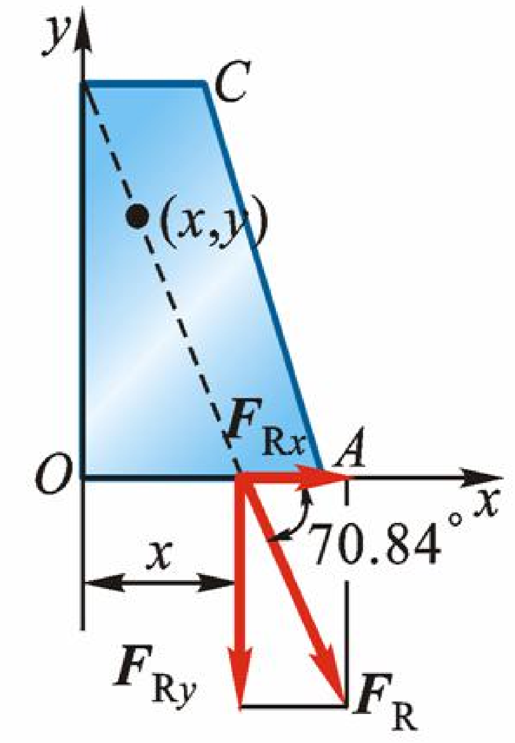
$$x=\frac{d}{\cos\left(90^0-70.84^0\right)}=3.514\text{m}$$

（3）求合力作用线方程：

$$M_O = \sum M_O\left(\vec{F}_{\mathrm{R}}\right) = x \cdot F_{\mathrm{R}y} - y \cdot F_{\mathrm{R}x} = x \cdot F'_{\mathrm{R}y} - y \cdot F'_{\mathrm{R}x}$$

$$\Rightarrow -2355 = x(-670.1) - y(232.9)$$

$$\Rightarrow 607.1x - 232.9y - 2355 = 0$$



§2-4 平面任意力系的平衡条件和平衡方程

一. 平面任意力系的平衡方程

平面任意力系平衡的充要条件是：

$$F'_{\mathrm{R}} = 0$$

力系的主矢和对任意点的主矩都等于零

$$M_O = 0$$

因为 $F'_{\mathrm{R}} = \sqrt{(\sum F_x)^2 + (\sum F_y)^2}$ $\qquad M_O = \sum M_O(\overline{F}_i)$

平面任意力系的平衡方程

$$\begin{cases} \sum F_x = 0 \\ \sum F_y = 0 \\ \sum M_O = 0 \end{cases}$$

一般式

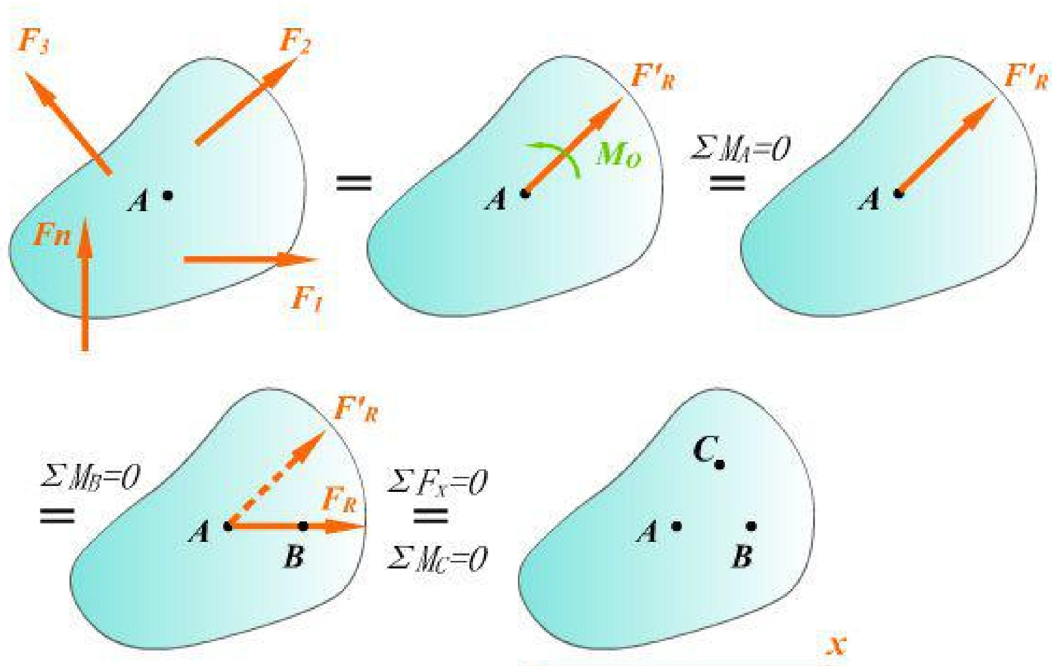
平面任意力系平衡的解析条件是：
所有各力在两个任选的坐标轴上
的投影的代数和分别等于零，
以

平面任意力系的平衡方程另两种形式

$$\begin{cases} \sum F_x = 0 \\ \sum M_A = 0 \\ \sum M_B = 0 \end{cases}$$

二矩式

两个取矩点连线，不得与投影轴垂直

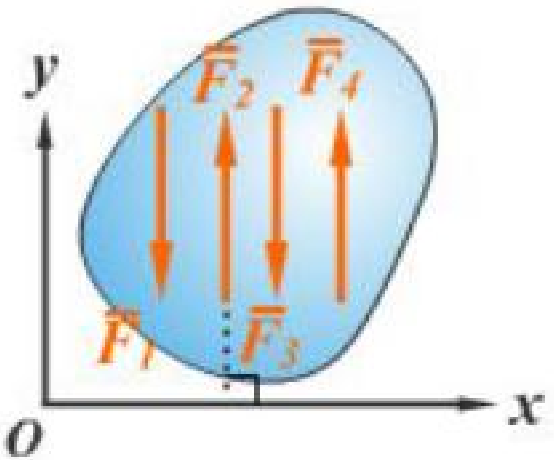


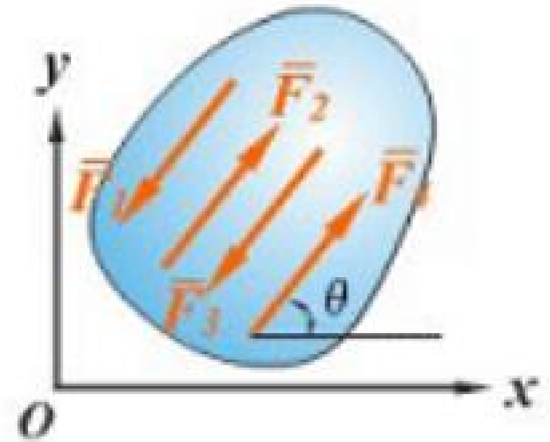
$$\begin{cases} \sum M_A = 0 \\ \sum M_B = 0 \\ \sum M_C = 0 \end{cases}$$

三矩式

三个取矩点，不得共线

二. 平面平行力系的平衡方程





$$\sum F_x = 0 \qquad 0+0+0+\cdots=0$$

$$\sum F_x = 0 \qquad F_1\cos\theta - F_2\cos\theta + F_3\cos\theta + \cdots = 0$$

$$\sum F_y = 0 \qquad F_1\sin\theta - F_2\sin\theta + F_3\sin\theta + \cdots = 0$$

$$\boxed{\begin{cases}\sum F_y = 0\\ \sum M_A = 0\end{cases}}$$

各力不得与投影轴垂直

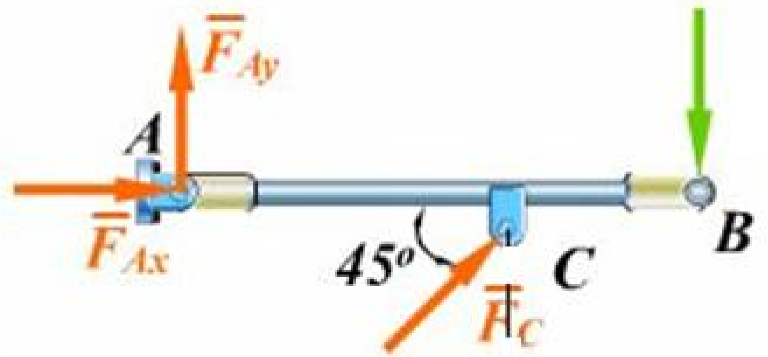
$$\boxed{\begin{cases}\sum M_A = 0\\ \sum M_B = 0\end{cases}}$$

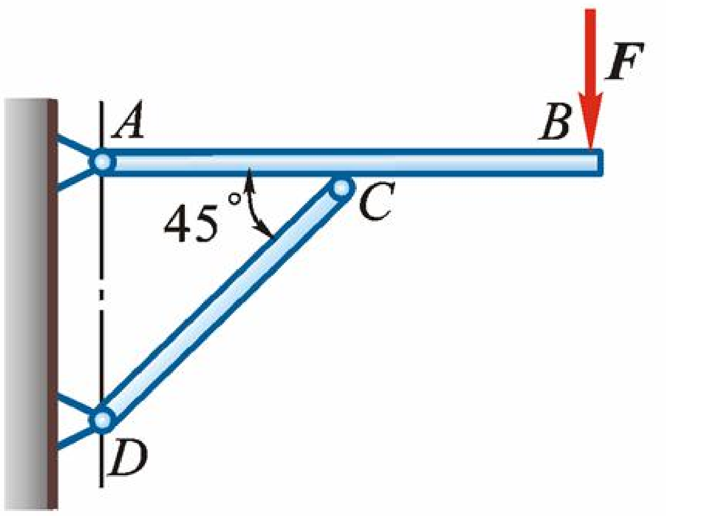
两点连线不得与各力平行

例2-12

已知：$AC = CB = l, F = 10\,\text{kN}$

求： 铰链 A 和 DC 杆受力.





解： 取 AB梁，画受力图.

$$\sum F_x = 0 \quad F_{Ax} + F_C \cos 45^\circ = 0$$

$$\sum F_y = 0 \quad F_{Ay} + F_C \sin 45^\circ - F = 0$$

$$\sum M_A = 0 \quad F_C \cos 45^\circ \cdot l - F \cdot 2l = 0$$

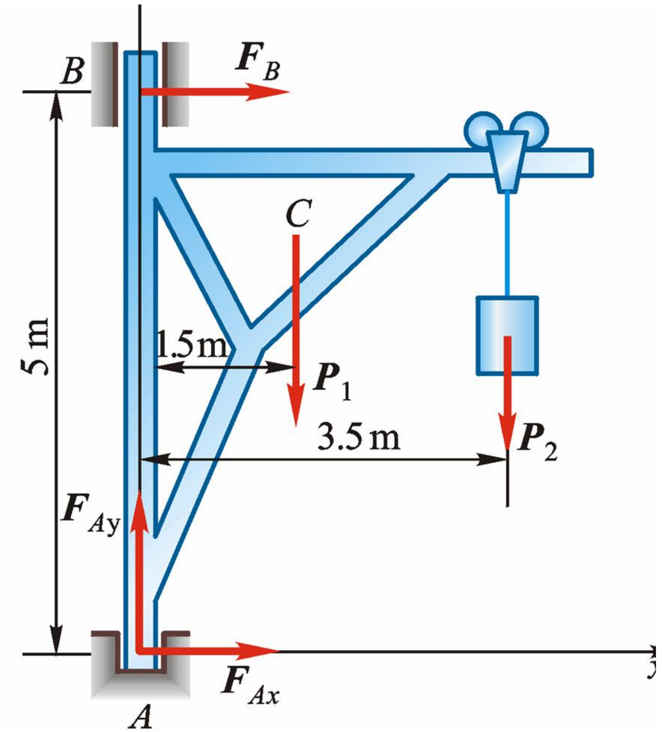
$$\Rightarrow \quad F_C = 28.28\text{kN}, F_{Ax} = -20\text{kN}, F_{Ay} = -10\text{kN}$$

例2-13

已知： $P_1 = 10\text{kN},\ P_2 = 40\text{kN}$，尺寸如图。

求： 轴承 A, B 处的约束力.

解： 取起重机，画受力图.



$$\sum F_x = 0 \qquad F_{Ax} + F_B = 0$$

$$\sum F_y = 0 \qquad F_{Ay} - P_1 - P_2 = 0$$

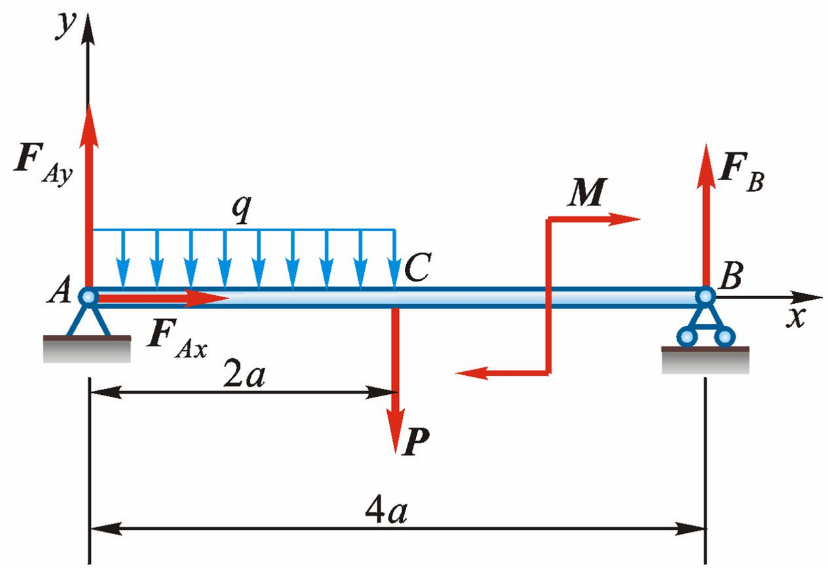
$$\sum M_A = 0 \qquad -F_B \cdot 5 - 1.5 \cdot P_1 - 3.5 \cdot P_2 = 0$$

$$F_{Ay} = 50\text{kN} \qquad F_B = -31\text{kN} \qquad F_{Ax} = 31\text{kN}$$

例2-14

已知：$P, q, a, M = qa$ 。

求：　支座 A, B 处的约束力.



解：取 AB梁，画受力图.

$$\sum F_x = 0 \qquad F_{Ax} = 0$$

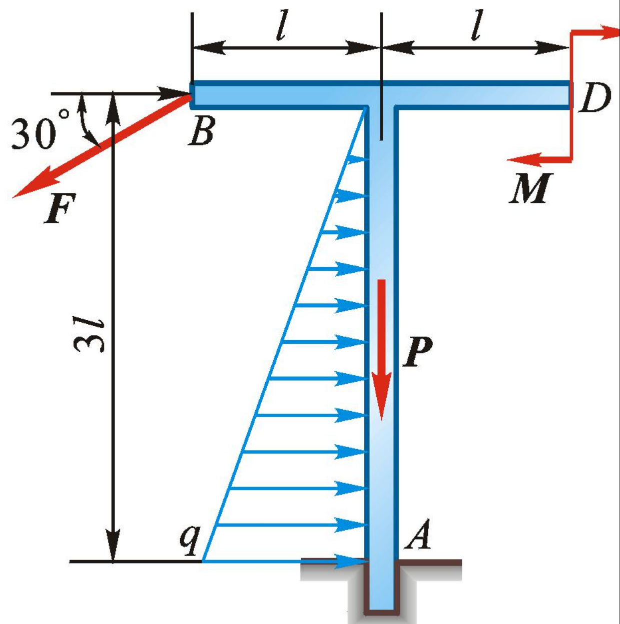
$$\sum M_A = 0 \quad F_B \cdot 4a - M - P \cdot 2a - q \cdot 2a \cdot a = 0$$

$$F_B = \frac{3}{4}P + \frac{1}{2}qa$$

$$\sum F_y = 0 \qquad F_{Ay} - q \cdot 2a - P + F_B = 0$$

$$F_{Ay} = \frac{P}{4} + \frac{3}{2}qa$$

例2—15

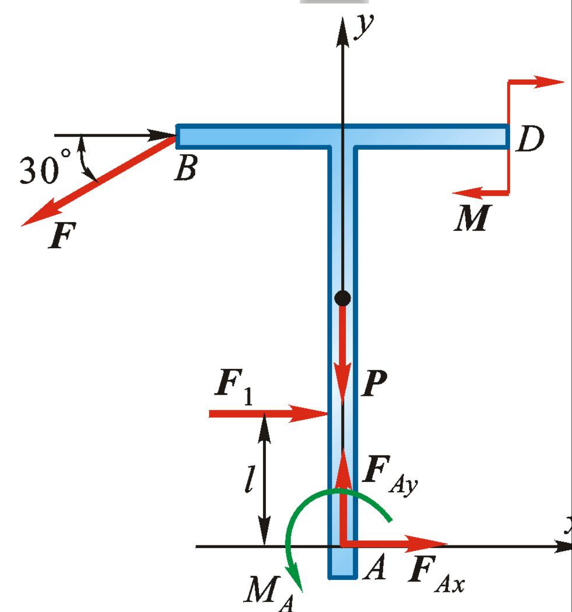


已知：$P=100\text{kN}, M=20\text{kN}\cdot\text{m},$

$q=20\text{kN/m}, F=400\text{kN}, l=1\text{m}$

求： 固定端 A 处约束力.

解： 取 T 型刚架，画受力图.



其中 $F_1=\dfrac{1}{2}q\times 3l=30\textbf{kN}$

$\sum F_x=0 \quad F_{Ax}+F_1-F\sin 60^\circ=0$

$\sum F_y=0 \quad F_{Ay}-P-F\cos 60^\circ=0$

$\sum M_A=0$

$M_A-M-F_1\cdot l+F\cos 60^\circ\cdot l+F\sin 60^\circ\cdot 3l=0$

$\Rightarrow \quad F_{Ax}=316.4\text{kN} \qquad F_{Ay}=300\text{kN}$

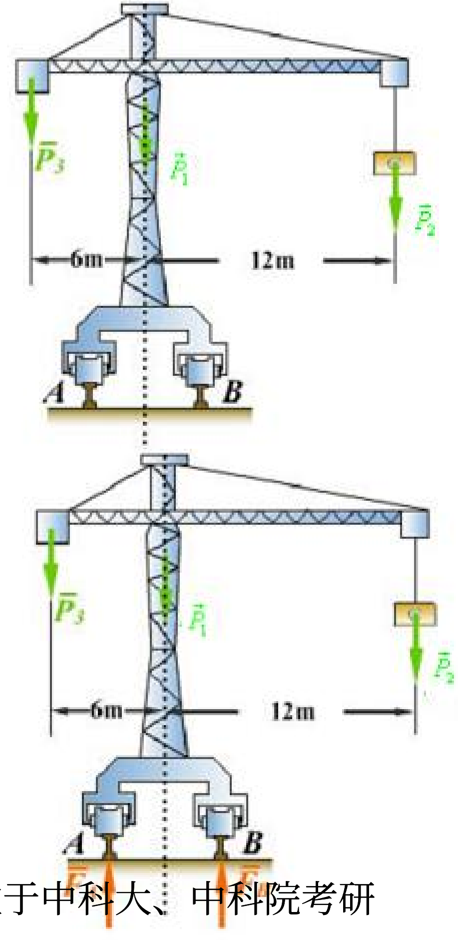
$M_A=-1188\text{kN}\cdot\text{m}$

例2-16 已知：$P_1 = 700\text{kN}, P_2 = 200\text{kN}, AB = 4\text{m}$

求：（1）起重机满载和空载时不翻倒，平衡载重 P_3；

（2）$P_3 = 180\text{kN}$，轨道 AB给起重机轮子的约束力。

解： 取起重机，画受力图.



满载时，$\vec{F}_A = 0,$

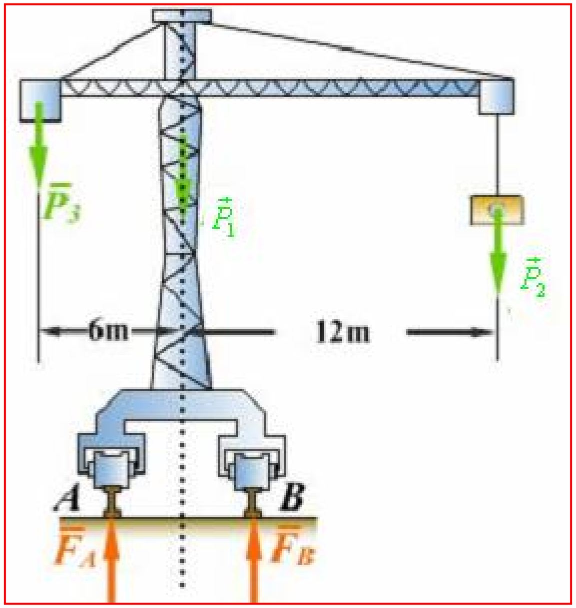
为不安全状况

$$\sum M_B = 0$$

$$P_{3\min} \cdot 8 + 2P_1 - 10P_2 = 0$$

$$\Rightarrow \quad P_{3\min} = 75\text{kN}$$

空载时， $\vec{F}_B = 0$， 为不安全状况



$$\sum M_A = 0 \qquad 4P_{3\max} - 2P_1 = 0$$

$$F_{3\max} = 350\text{kN}$$

$$75\text{kN} \le P_3 \le 350\text{kN}$$

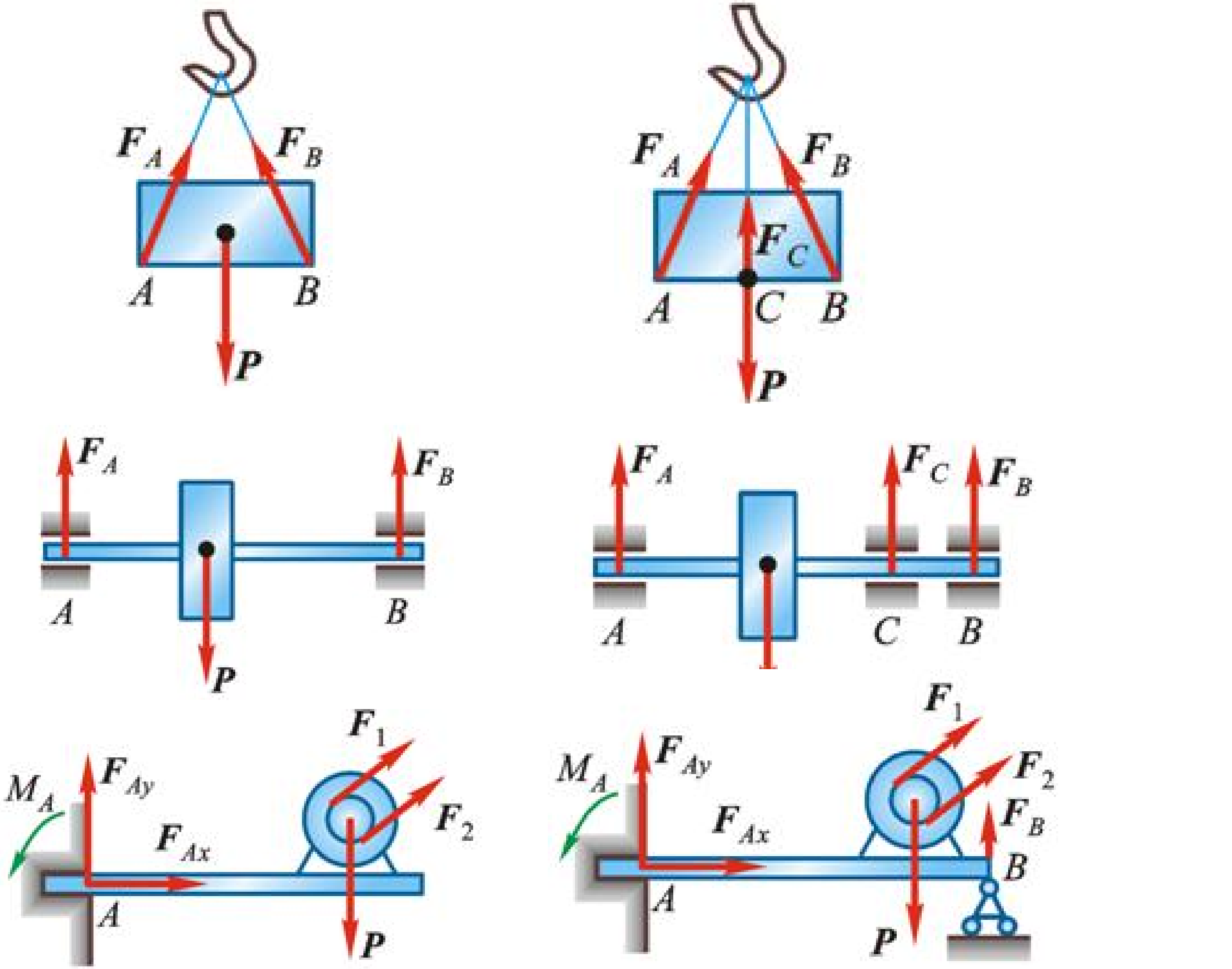
$P_3 = 180\text{kN}$ 时

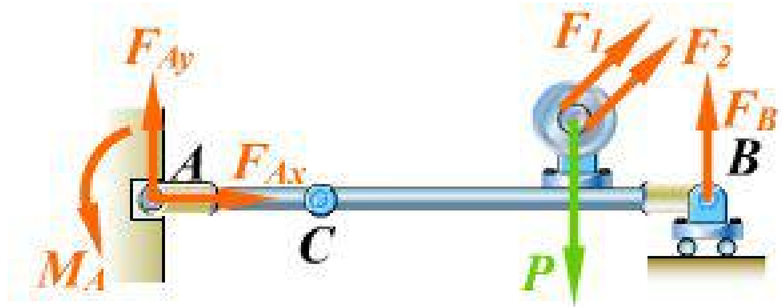
$$\sum M_A = 0 \quad 4P_3 - 2P_1 - 14P_2 + 4F_B = 0$$

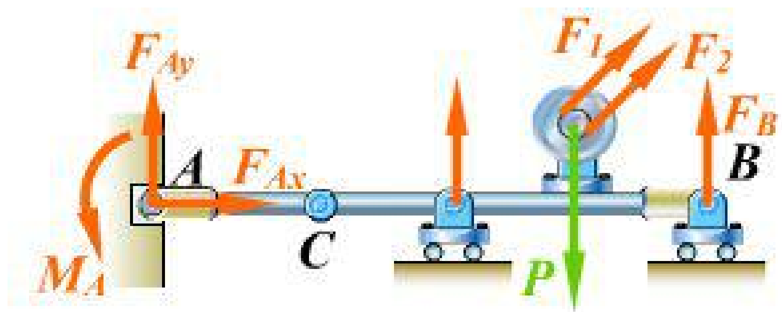
$$\sum F_{iy} = 0 \quad F_A + F_B - P_1 - P_2 - P_3 = 0$$

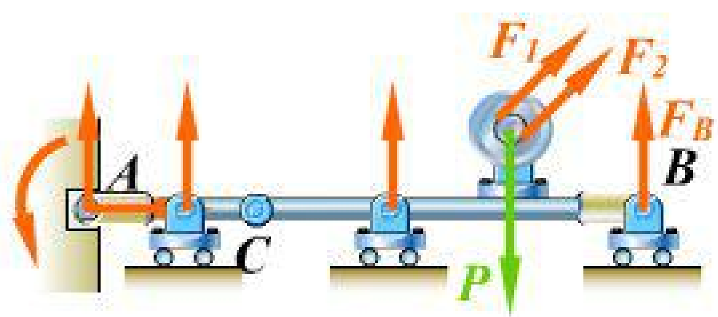
$$F_A = 210\text{kN} \qquad F_B = 870\text{kN}$$

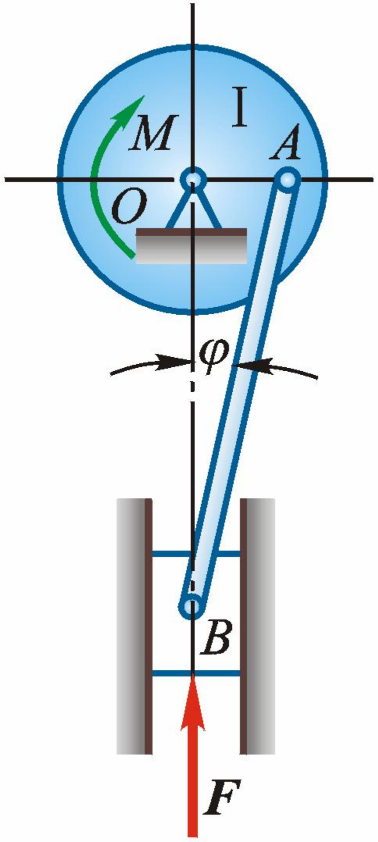
§2-5 物体系的平衡·静定和超静定问题




F_{Ay}
F_1
F_2
F_B
B
A
F_{Ax}
C
P
M_A


F_{Ay}
F_1
F_2
F_B
B
A
F_{Ax}
C
P
M_A


F_1
F_2
F_B
B
A
C
P



例2-17 已知：$OA=R, AB=l, \vec{F}$，不计物体自重与摩擦，系统在图示位置平衡；

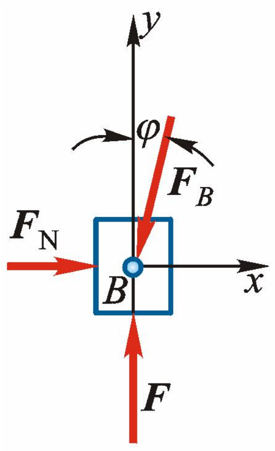
求：力偶矩M的大小，轴承O处的约束力，连杆AB受力，冲头给导轨的侧压力.

解： 取冲头B，画受力图.

$$\sum F_y = 0 \qquad F - F_B \cos\varphi = 0$$

$$\sum F_x = 0 \qquad F_N - F_B \sin\varphi = 0$$

$$F_B = \frac{F}{\cos\varphi} = \frac{Fl}{\sqrt{l^2 - R^2}} \qquad F_N = F\tan\varphi = \frac{FR}{\sqrt{l^2 - R^2}}$$



取轮，画受力图.

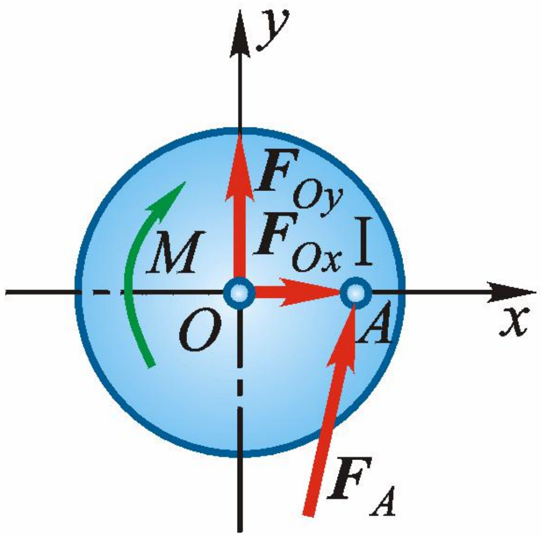
$$\sum F_x = 0 \qquad F_{Ox} + F_A \cos\varphi = 0$$

$$\sum F_y = 0 \qquad F_{Oy} + F_A \sin\varphi = 0$$

$$\sum M_O = 0 \qquad F_A \cos\varphi - M = 0$$

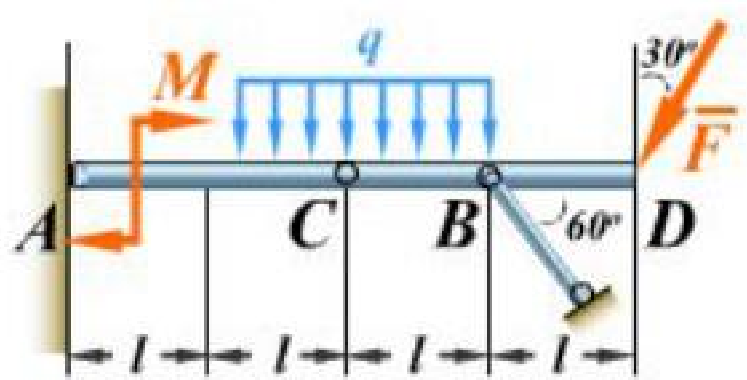
$$F_{Ox} = -\frac{FR}{\sqrt{l^2 - R^2}} \qquad F_{Oy} = -F$$

$$M = FR$$



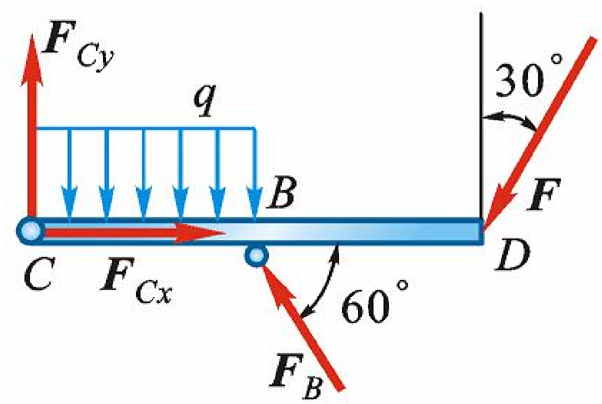
例2-18 已知： F=20kN, q=10kN/m, $M=20\text{kN}\cdot\text{m}$, l=1m；

求： ***A*,*B*处的约束力.**



解： 取***CD***梁, 画受力图.

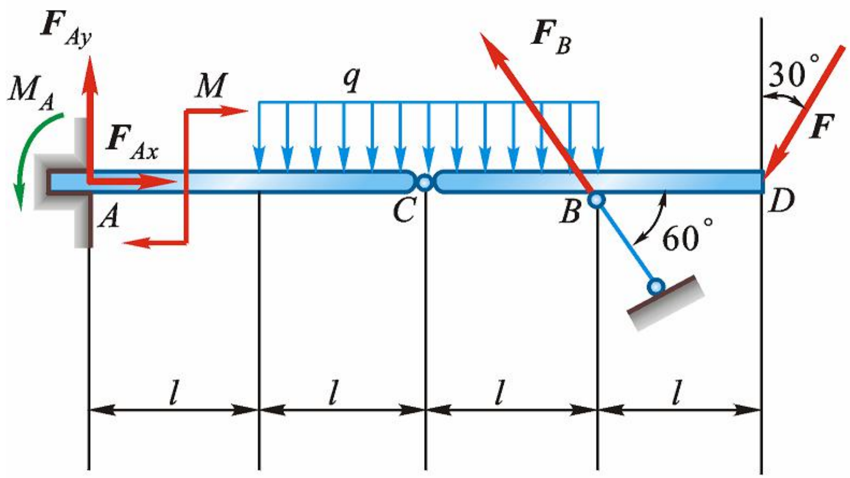
$$\sum M_C = 0$$



$$F_B \sin 60^\circ \cdot l - ql \cdot \frac{l}{2} - F\cos 30^\circ \cdot 2l = 0$$

F_B=45.77kN

取整体，画受力图.



$$\sum F_x = 0$$

$$F_{Ax} - F_B\cos 60^\circ - F\sin 30^\circ = 0$$

$$\sum F_y = 0$$

$$F_{Ay} - F_B\sin 60^\circ - 2ql - F\cos 30^\circ = 0$$

$$\sum M_A = 0$$

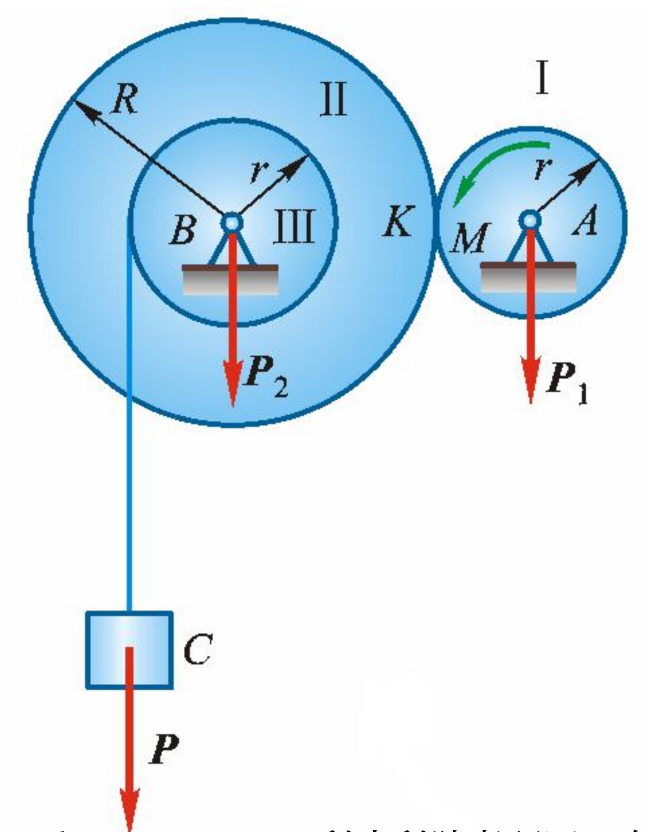
$$M_A - M - 2ql\cdot 2l + F_B\sin 60^\circ\cdot 3l - F\cos 30^\circ\cdot 4l = 0$$

$$\Rightarrow \quad M_A = 10.37\text{kN}\cdot\text{m} \qquad F_{Ax} = 32.89\text{kN} \qquad F_{Ay} = -2.32\text{kN}$$

例2-19

已知：$P_2=2P_1$，$P=20P_1$，r，$R=2r$，$\alpha=20^\circ$；

求：物C匀速上升时，作用于小轮上的力偶矩M，
轴承A，B处的约束力.



解： 取塔轮及重物 C，画受力图.

$$\sum M_B = 0 \quad F_t \cdot R - P \cdot r = 0 \qquad F_t = \frac{P \cdot r}{R} = 10P_1$$

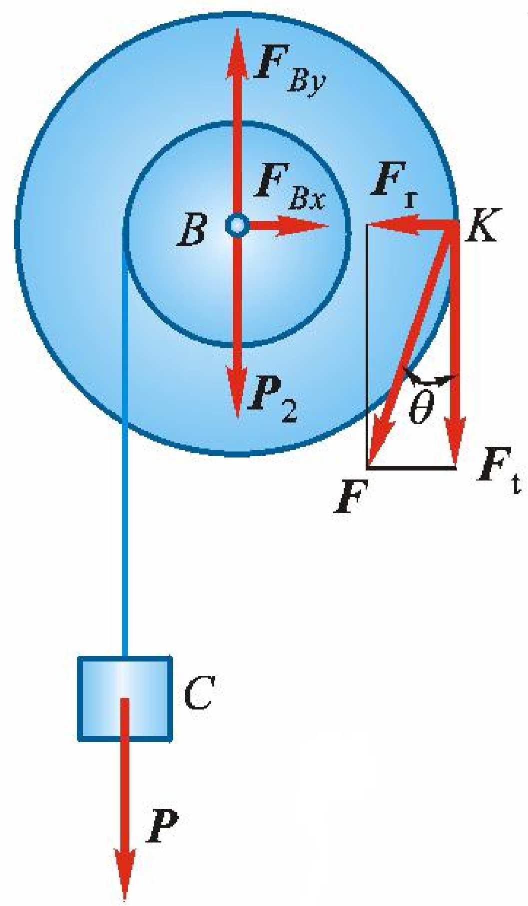
由 $\dfrac{F_r}{F_t} = \tan 20^\circ$

$$F_r = F_t \cdot \tan 20^\circ = 3.64P_1$$

$$\sum F_x = 0 \qquad F_{Bx} - F_r = 0$$

$$\sum F_y = 0 \qquad F_{By} - P - P_2 - F_t = 0$$

$$F_{Bx} = 3.64P_1 \qquad F_{By} = 32P_1$$

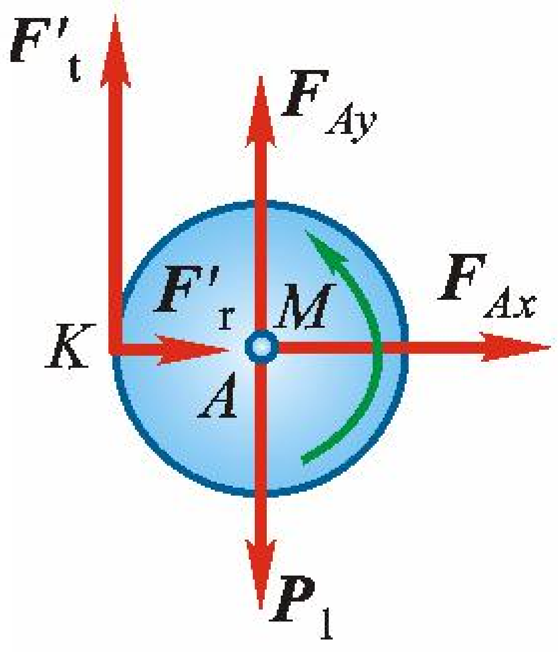


取小轮，画受力图.

$$\sum F_x = 0 \qquad F_{Ax} + F_r' = 0$$

$$\sum F_y = 0 \qquad F_{Ay} + F_t' - P_1 = 0$$

$$\sum M_A = 0 \qquad M - F_t' \cdot r = 0$$



$$\boldsymbol{F}_{Ax} = -3.64\boldsymbol{P}_1$$

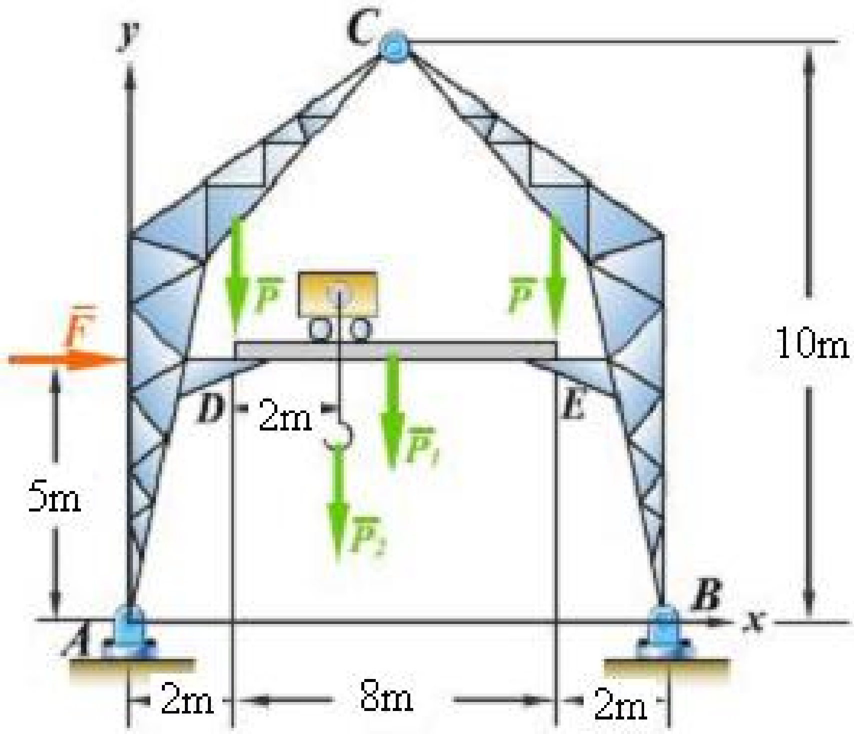
$$\boldsymbol{F}_{Ay} = -9\boldsymbol{P}_1$$

$$M = 10 P_1 r$$

例2-20

已知：P=60kN, P_1=20kN, P_2=10kN, 风载F=10kN,
尺寸如图；

求：***A,B***处的约束力.



解: 取整体, 画受力图.

$$\sum M_A = 0 \quad 12F_{By} - 10P - 6P_1 - 4P_2 - 2P - 5F = 0$$

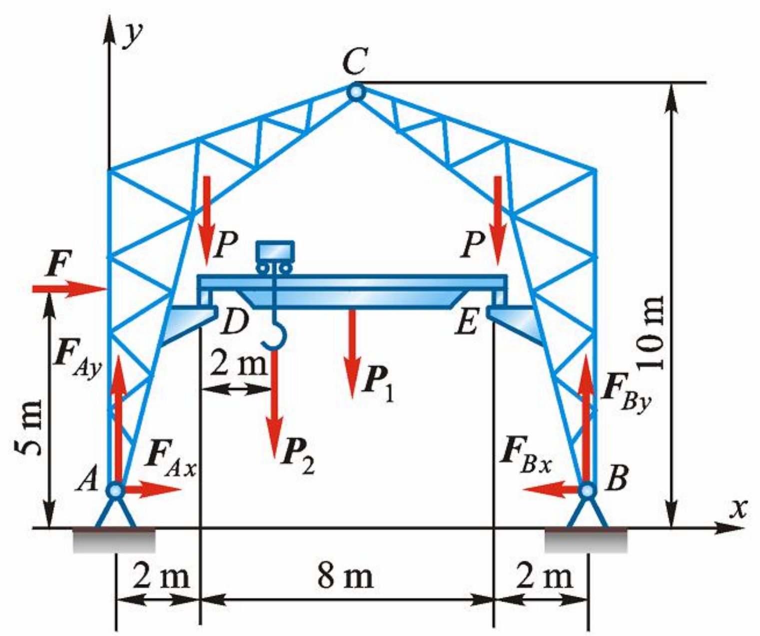
$$\sum F_y = 0 \quad F_{Ay} + F_{By} - 2P - P_1 - P_2 = 0$$

$$\sum F_x = 0 \quad F_{Ax} + F - F_{Bx} = 0$$

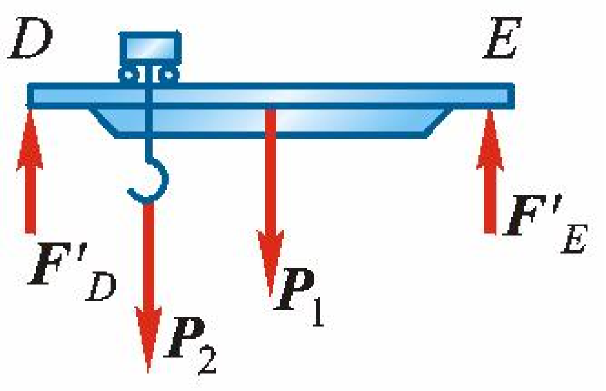
$$F_{Ay} = 72.5\text{kN}$$

$$F_{By} = 77.5\text{kN}$$

$$F_{Ax} = F_{Bx} - F$$



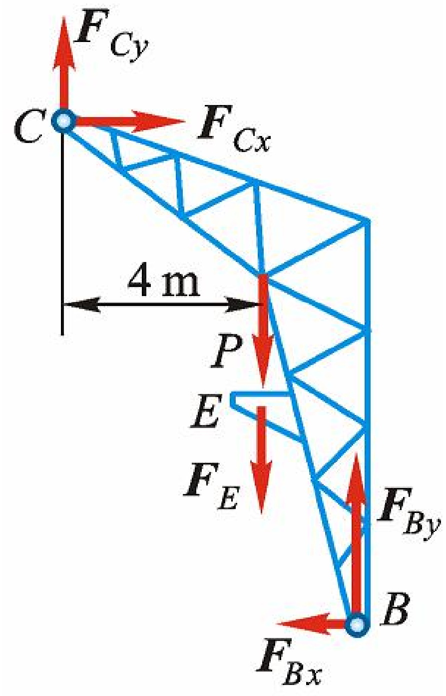
取吊车梁，画受力图.



$$\sum M_D = 0 \qquad 8F_E' - 4P_1 - 2P_2 = 0$$

$$\Rightarrow \quad F_E' = 12.5\text{kN}$$

取右边刚架，画受力图.



$$\sum M_C = 0$$

$$6F_{By} - 10F_{Bx} - 4(P + F_E) = 0$$

$$\Rightarrow \quad F_{Bx} = 17.5\text{kN}$$

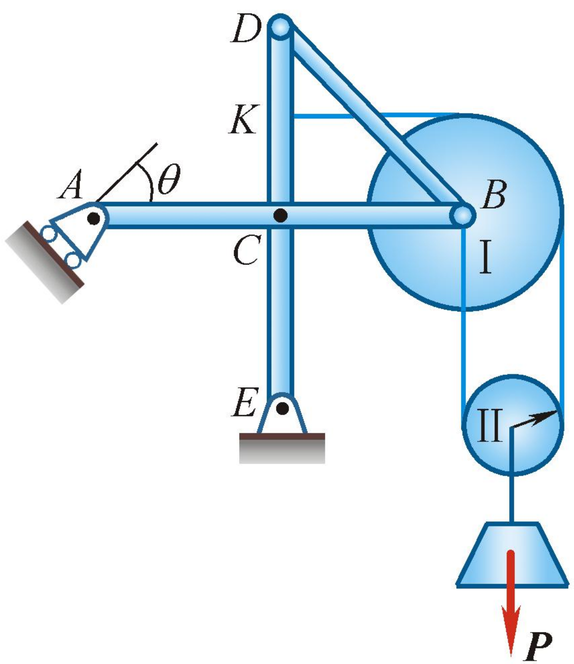
$$F_{Ax} = 7.5\text{kN}$$

例2-21

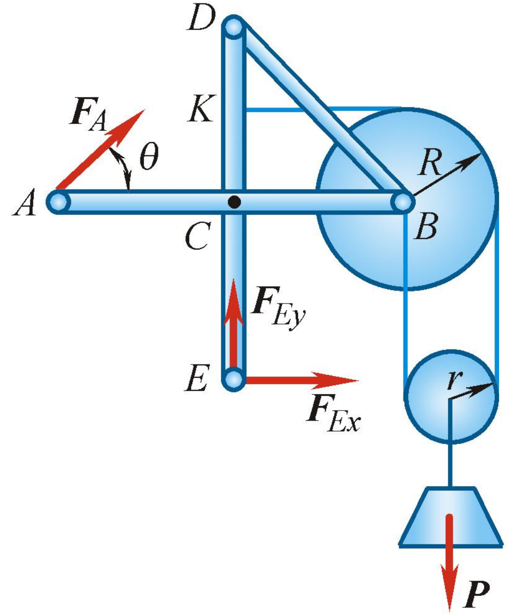
已知：$DC=CE=CA=CB=2l,\ R=2r=l,\ \vec{P}$，各构件自重不计，

$$\theta = 45^0.$$

求：***A*, *E*** 支座处约束力及 ***BD*** 杆受力.



解： 取整体，画受力图.



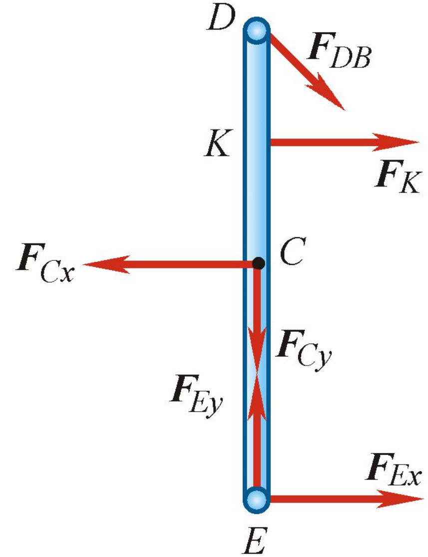
$$\sum M_E = 0 \qquad -F_A \cdot \sqrt{2} \cdot 2l - P \cdot \frac{5}{2} l = 0$$

$$\sum F_x = 0 \qquad F_{Ex} + F_A \cos 45^0 = 0$$

$$\sum F_y = 0 \qquad F_{Ey} - P + F_A \sin 45^0 = 0$$

$$\Rightarrow \quad F_A = -\frac{5\sqrt{2}}{8} P \quad F_{Ex} = \frac{5}{8} P \quad F_{Ey} = \frac{13}{8} P$$

取*DCE*杆，画受力图.



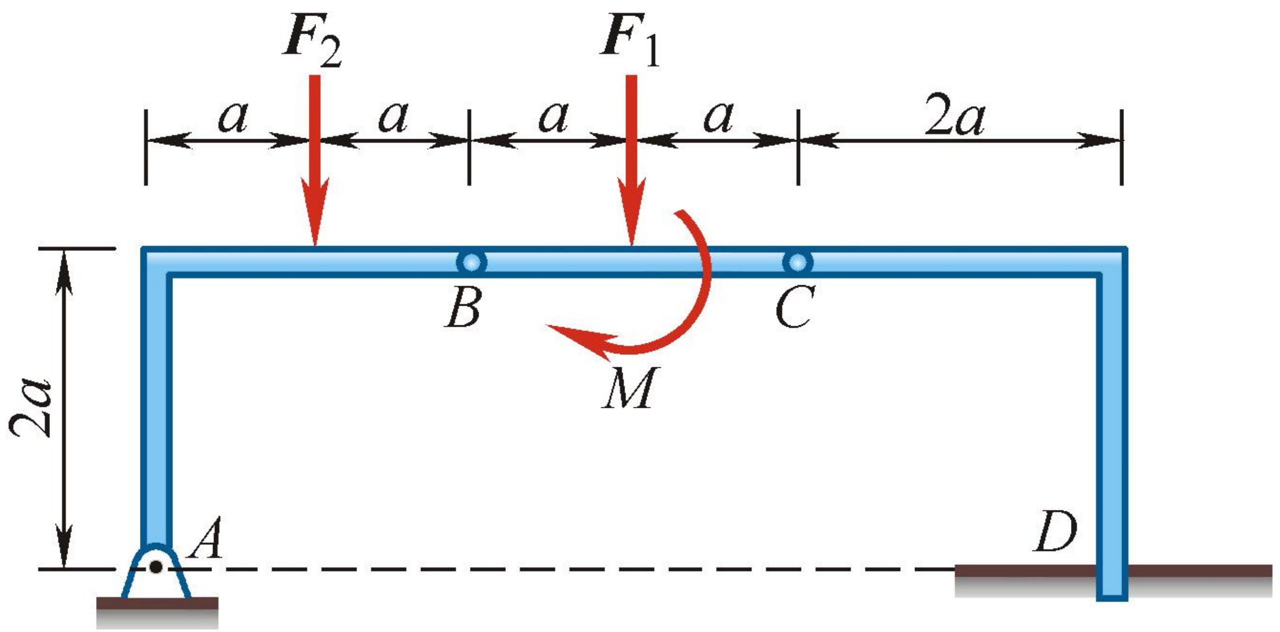
$$\sum M_C = 0 \quad -F_{DB} \cos 45^0 \cdot 2l - F_K \cdot l + F_{Ex} \cdot 2l = 0$$

$$F_{DB} = \frac{3\sqrt{2}}{8} P \qquad \text{（拉）}$$

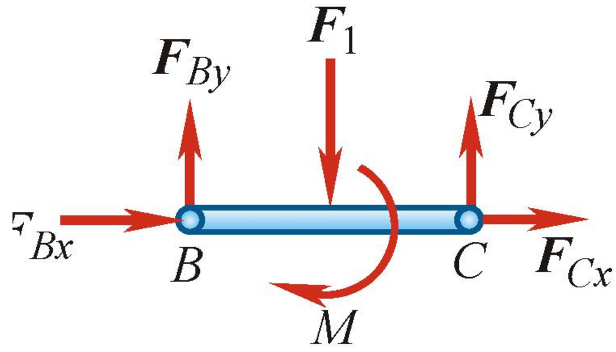
例2-22

已知：如图所示结构，a，$M = Fa$ ，$F_1 = F_2 = F$ ．

求：A，D处约束力.



解： 以*BC*为研究对象，受力如图所示.

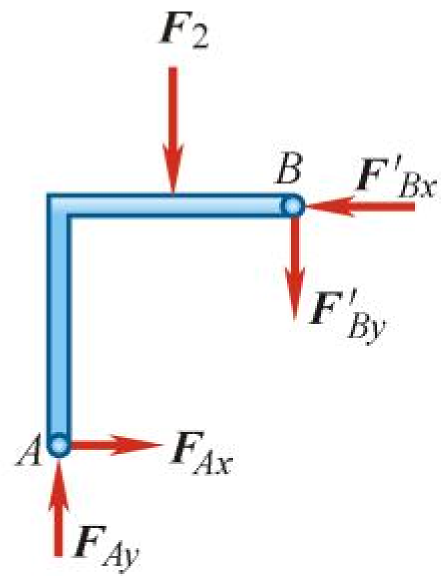


$$\sum M_B = 0 \qquad F_{Cy} \cdot 2a - F_1 a - M = 0$$

$$\sum F_y = 0 \qquad F_{By} + F_{Cy} - F_1 = 0$$

$$\Rightarrow \qquad F_{Cy} = F \qquad F_{By} = 0$$

以*AB*为研究对象，受力如图所示.



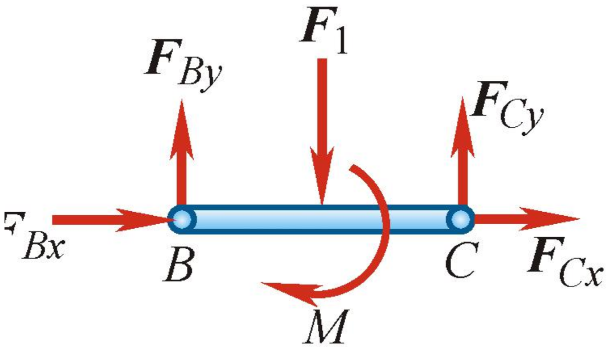
$$\sum M_A = 0 \quad F_{Bx}' \cdot 2a - F_{By}' \cdot 2a - F_2 a = 0$$

$$\sum F_x = 0 \qquad F_{Ax} - F_{Bx}' = 0$$

$$\sum F_y = 0 \qquad F_{Ay} - F_{By}' - F_2 = 0$$

$$\Rightarrow \qquad F_{Bx}' = F_{Ax} = \frac{1}{2} F \qquad F_{Ay} = F$$

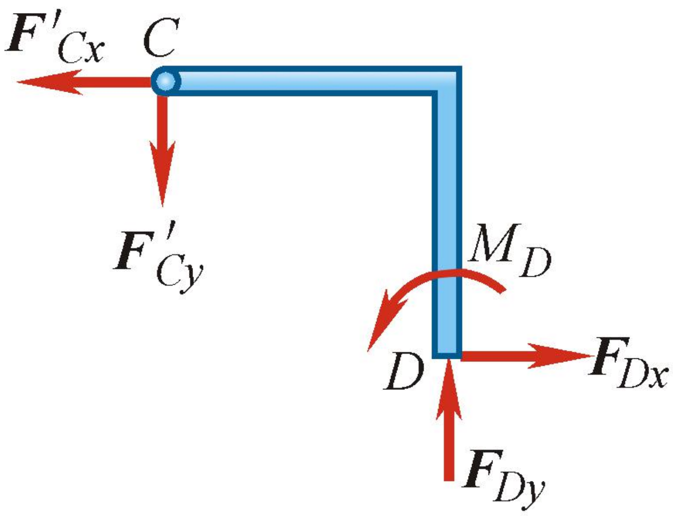
再分析*BC*.



$$\sum F_x = 0 \qquad F_{Cx} + F_{Bx} = 0$$

$$\Rightarrow \qquad F_{Cx} = -\frac{1}{2}F$$

以*AB*为研究对象，受力如图所示.



$$\sum F_x = 0 \qquad F_{Dx} - F_{Cx}' = 0$$

$$\sum F_y = 0 \qquad F_{Dy} - F_{Cy}' = 0$$

$$\sum M_D = 0 \quad M_D + F_{Cy}' \cdot 2a + F_{Cx}' \cdot 2a = 0$$

$$\Rightarrow \qquad F_{Dx} = -\frac{1}{2}F \qquad F_{Dy} = F \quad M_D = -Fa$$

例 2-23　已知：P=10kN ，a ，杆、轮重不计；

求：　A ,C支座处约束力.

解：取整体，受力图能否这样画？

取整体，画受力图.

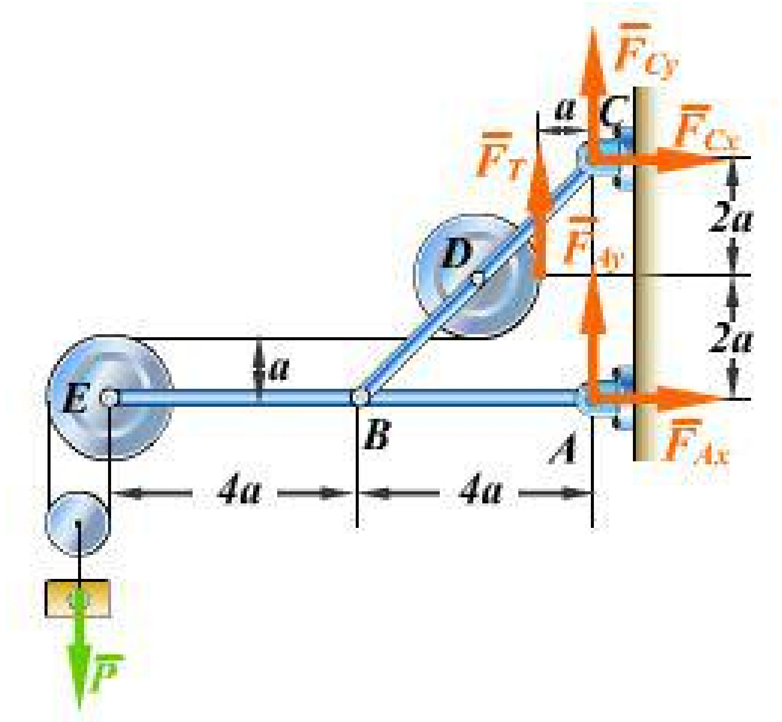
$$\sum M_C = 0 \quad 4aF_{Ax} + 8.5aP - F_T a = 0$$

$$F_{Ax} = -20\text{kN}$$

$$\sum F_x = 0 \quad F_{Ax} + F_{Cx} = 0$$

$$F_{Cx} = 20\text{kN}$$

$$\sum F_y = 0 \quad F_{Ay} + F_{Cy} + F_T - P = 0$$

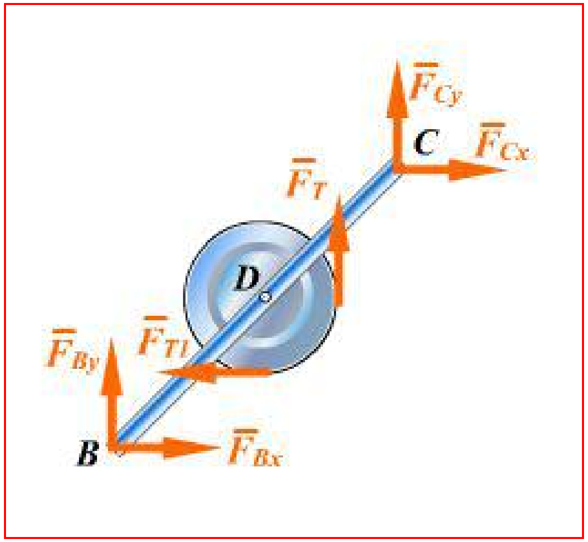


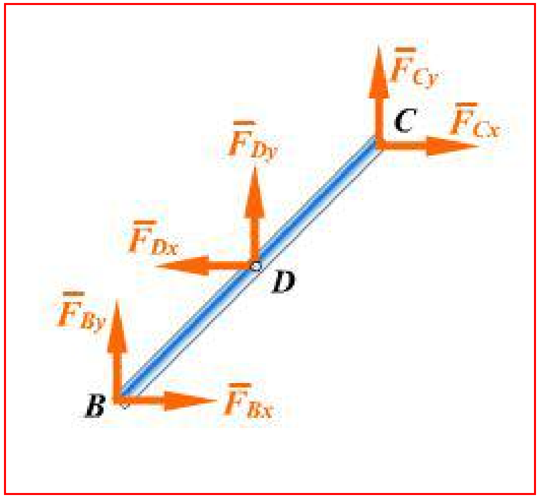
取*BDC*杆（带着轮）

$$\sum M_B = 0$$

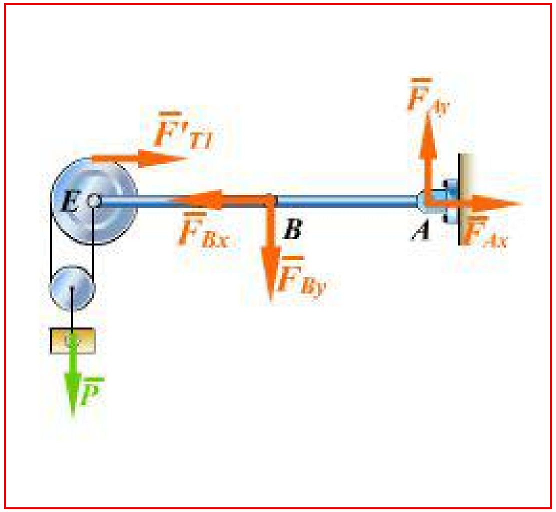
$$4aF_{Cy} + F_T \cdot 3a + F_{T1} \cdot a - F_{Cx} \cdot 4a = 0$$

$$F_{Cy} = 15\text{kN} \qquad F_{Ay} = -10\mathbf{kN}$$

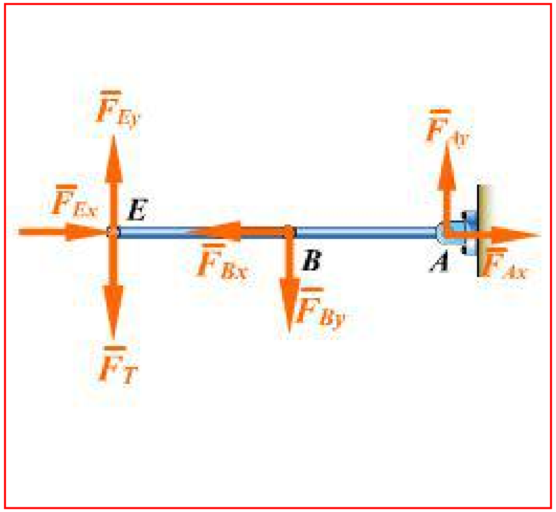




取*BDC* 杆（不带着轮）



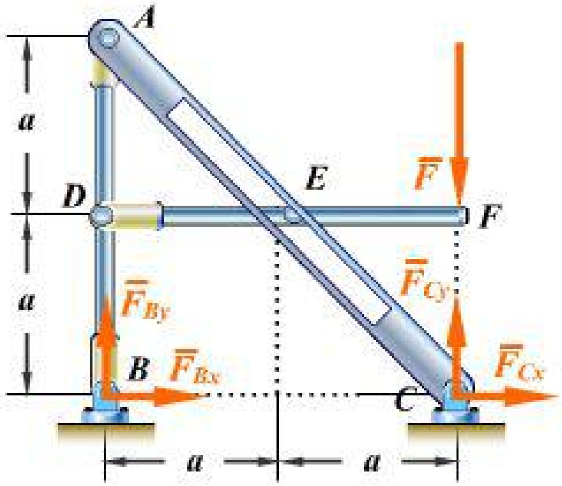
取*ABE*（带着轮）



取*ABE*杆（不带着轮）

例2-24 已知：P，a，各杆重不计；

求：B 铰处约束力.

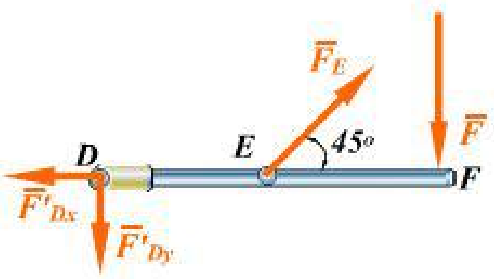


解： 取整体，画受力图

$$\sum M_C = 0 \qquad -F_{By} \cdot 2a = 0$$

$$\Rightarrow \qquad F_{By} = 0$$

取DEF杆，画受力图



$$\sum M_D = 0 \qquad F_E \sin 45^\circ \cdot a - F \cdot 2a = 0$$

$$\sum F_x = 0 \qquad F_E \cos 45^\circ - F_{Dx}' = 0$$

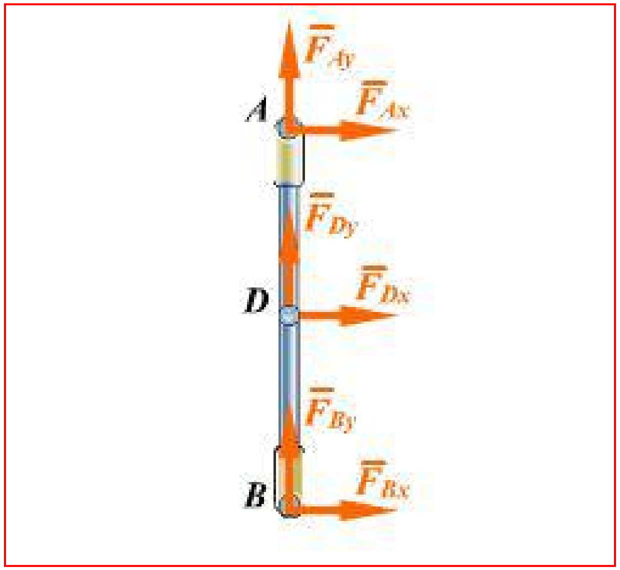
$$\sum M_E = 0 \qquad F_{Dy}' \cdot a - F \cdot 2a = 0$$

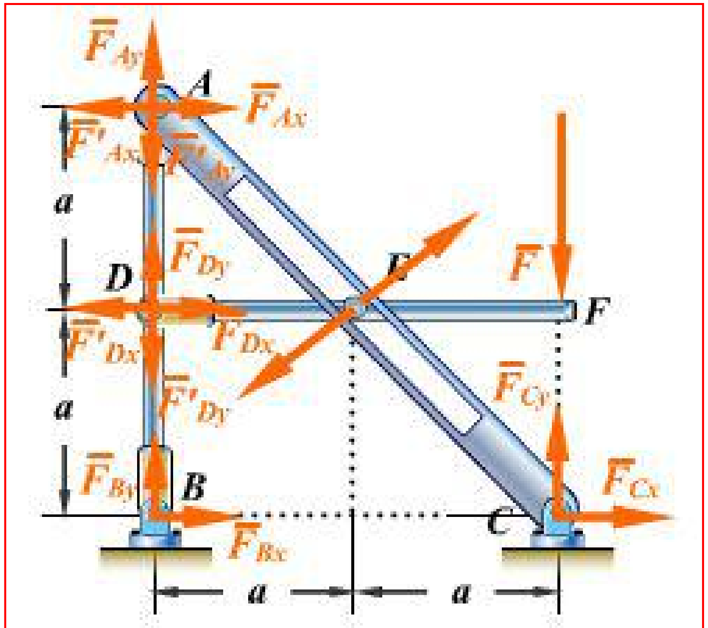
$$\Rightarrow \quad F_E \sin 45^\circ = 2F \qquad F_{Dx}' = F_E \cos 45^\circ = 2F \qquad F_{Dx}' = 2F$$

对*ADB*杆受力图

$$\sum M_A = 0 \qquad F_{Bx} \cdot 2a + F_{Dx} \cdot a = 0$$

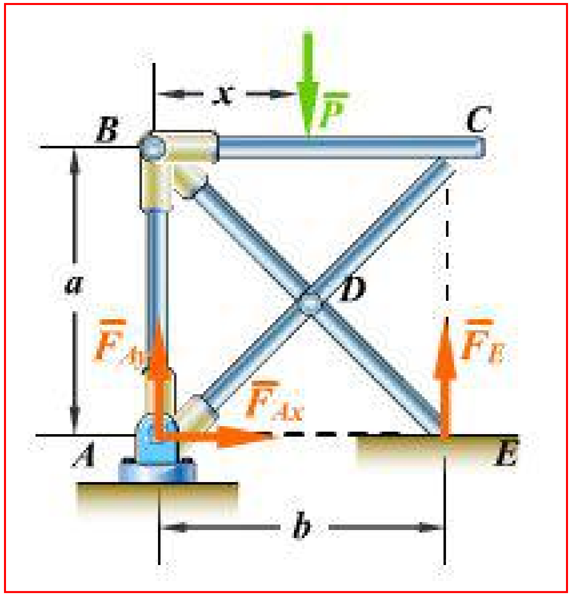
$$\Rightarrow \qquad F_{Bx} = -F$$





例2-25 已知： a ,b ,P,各杆重不计， C,E处光滑；

求证： AB杆始终受压，且大小为P.



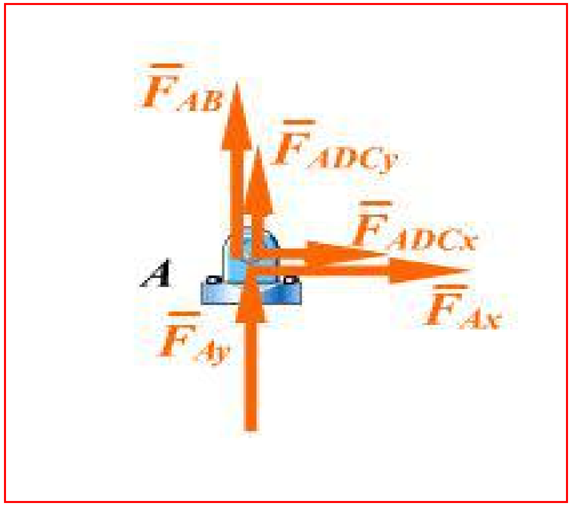
解： 取整体，画受力图.

$$\sum F_x = 0 \qquad F_{Ax} = 0$$

$$\sum M_E = 0 \qquad P\cdot(b-x) - F_{Ay}\cdot b = 0$$

$$\Rightarrow \quad F_{Ay} = \frac{P}{b}(b-x)$$

取销钉A，画受力图



$$\sum F_x = 0 \qquad F_{Ax} + F_{ADCx} = 0$$

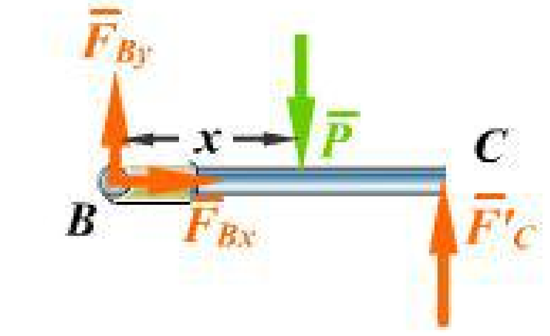
$$\sum F_y = 0 \qquad F_{AB} + F_{Ay} + F_{ADCy} = 0$$

$$\Rightarrow \qquad F_{ADCx} = 0$$

取*BC*，画受力图.

$$\sum M_B = 0 \qquad F_C' \cdot b - Px = 0$$

$$\Rightarrow \quad F_C' = \frac{x}{b}P$$

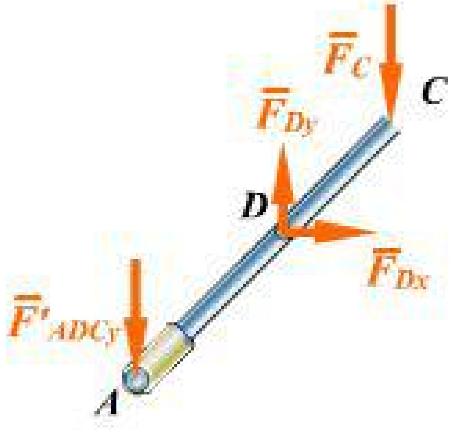


取*ADC*杆，画受力图.

$$\sum M_D = 0 \qquad F_{ADCy}' \cdot \frac{b}{2} - F_C \cdot \frac{b}{2} = 0$$

$$\Rightarrow \quad F_{ADCy}' = F_C = \frac{x}{b}P$$

$$\Rightarrow \quad F_{AB} = -P \text{（压）}$$

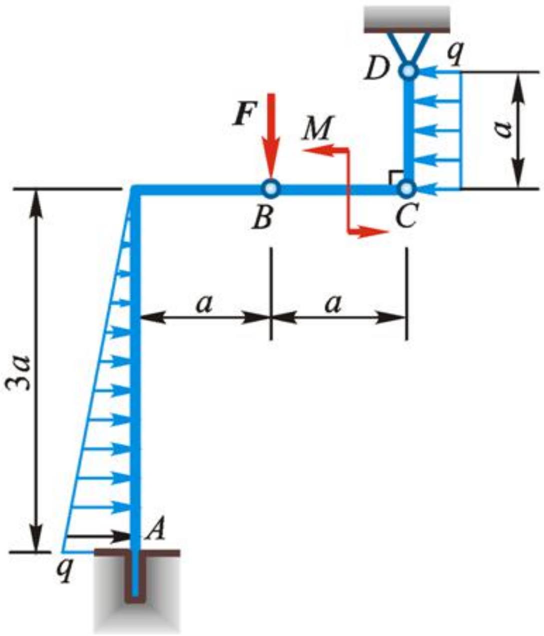


例2-26

已知：q ,a ,M ， 且$M=qa^2$，

P作用于销钉B上；

求： 固定端A处的约束力和销钉B对BC杆、AB杆的作用力.

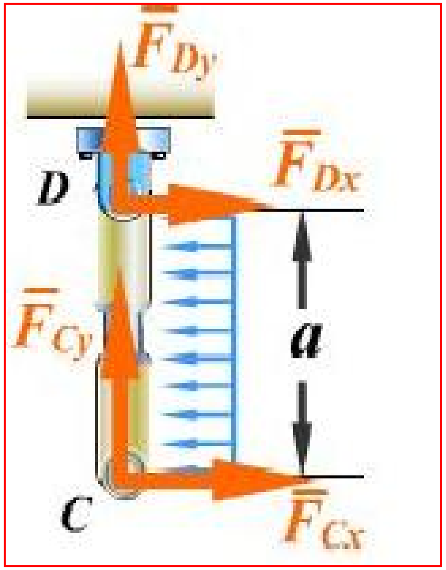


解： 取CD杆，画受力图.

$$\sum M_D=0$$

$$F_{Cx}\cdot a-qa\cdot\frac{a}{2}=0$$

$$\Rightarrow\quad F_{Cx}=\frac{1}{2}qa$$

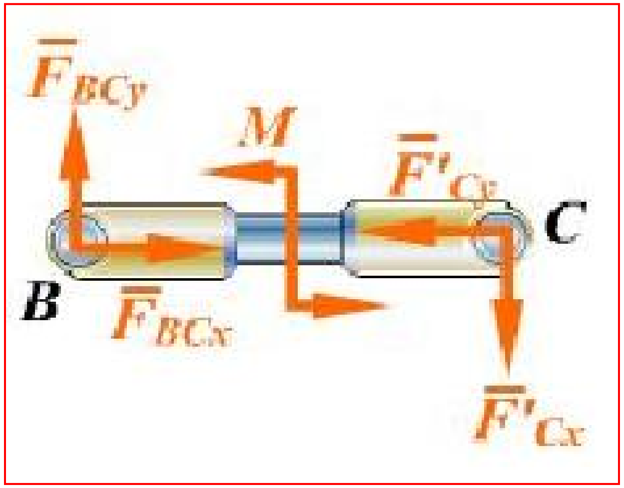


取BC杆（不含销钉B），画受力图.

$$\sum F_x = 0 \qquad F_{BCx} - F'_{Cx} = 0$$

$$\sum M_C = 0 \qquad M - F_{BCy} a = 0$$

$$\Rightarrow \quad F_{BCx} = \frac{1}{2} qa \qquad F_{BCy} = qa$$



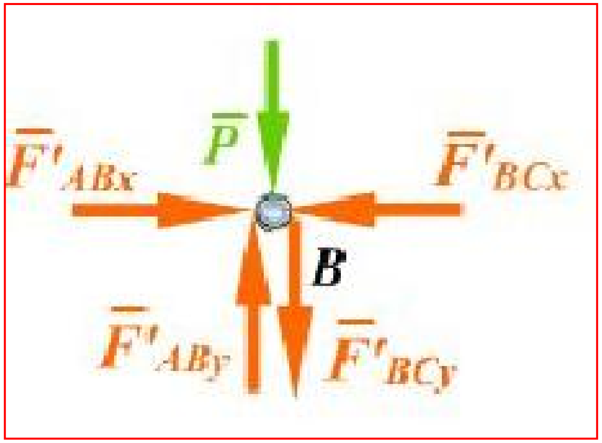
取销钉B，画受力图.

$$\sum F_x = 0 \qquad F'_{ABx} - F'_{BCx} = 0$$

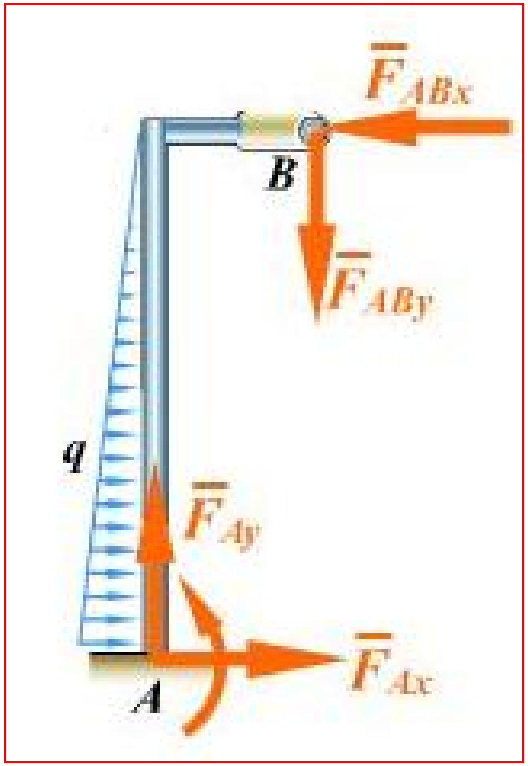
$$\sum F_y = 0 \qquad F'_{ABy} - F'_{BCy} - P = 0$$

$$\Rightarrow \quad F'_{ABx} = \frac{1}{2} qa \qquad F'_{ABy} = P + qa$$

$$\Rightarrow \quad F_{ABx} = -\frac{1}{2} qa \qquad F_{ABy} = -(P + qa)$$



取*AB*杆（不含销钉*B*），画受力图.



$$\sum F_x = 0 \qquad F_{Ax} + \frac{1}{2} \cdot q \cdot 3a - F_{ABx} = 0$$

$$\Rightarrow \qquad F_{Ax} = -qa$$

$$\sum F_y = 0 \qquad F_{Ay} - F_{ABy} = 0$$

$$\Rightarrow \qquad F_{Ay} = P + qa$$

$$\sum M_A = 0$$

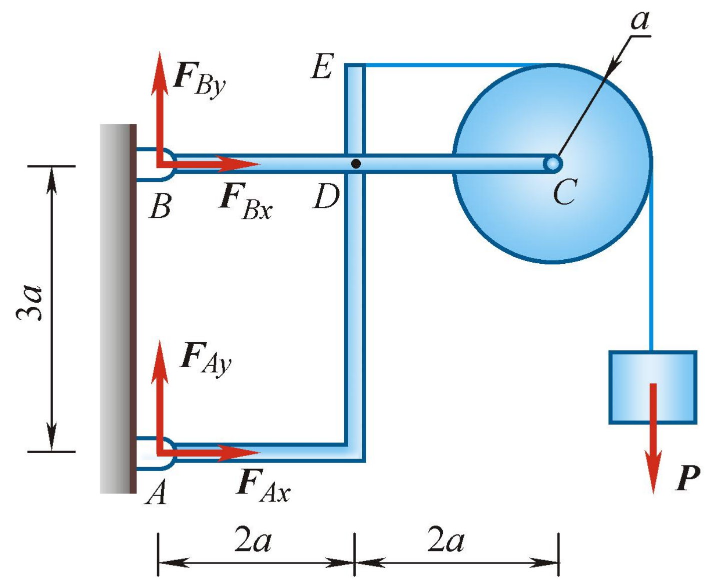
$$M_A - \frac{1}{2} \cdot q \cdot 3a \cdot a + F_{ABx} \cdot 3a - F_{ABy} \cdot a = 0$$

$$\Rightarrow \qquad M_A = (P + qa)a$$

思考-1

已知：如图所示结构，***P***和***a***.

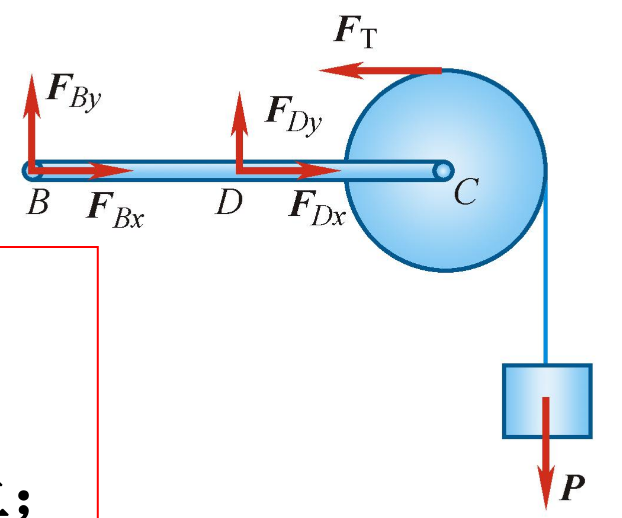
求：支座*A*，***B*** 处约束力.



解题思路：

先分析整体 ➡ F_{Bx} F_{Ax}

再分析***BC*** ➡ F_{By} F_{Ay}



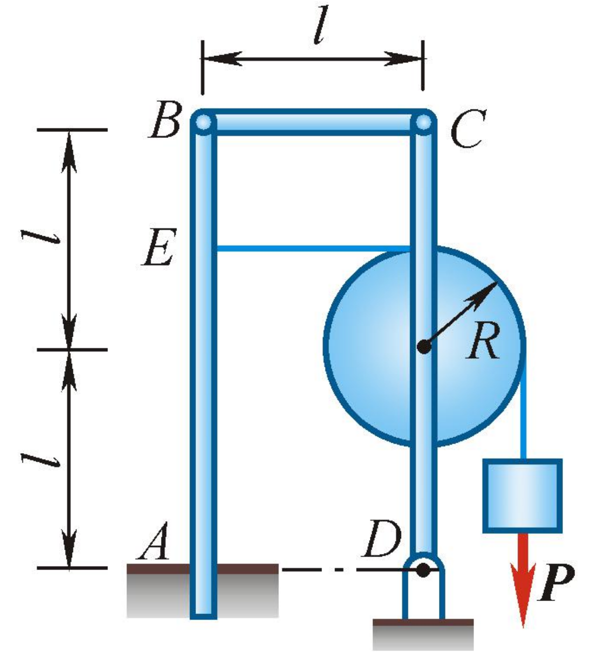
总结：

- 一般先分析整体；
- 一般不拆滑轮；
- 矩心尽量取在较多未知力的交点上；
- 投影轴尽量与较多未知力相垂直。

思考-2

已知：如图所示结构，***P***，***l***，***R***.

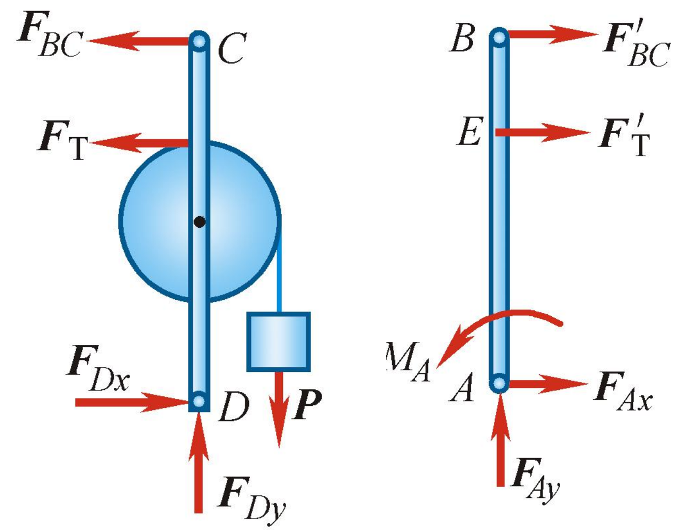
求：固定端*A*处约束力.



解题思路：

先分析杆***CD*** ➡ F_{BC}

再分析杆***AB*** ➡ F_{Ax} F_{Ay} M_A



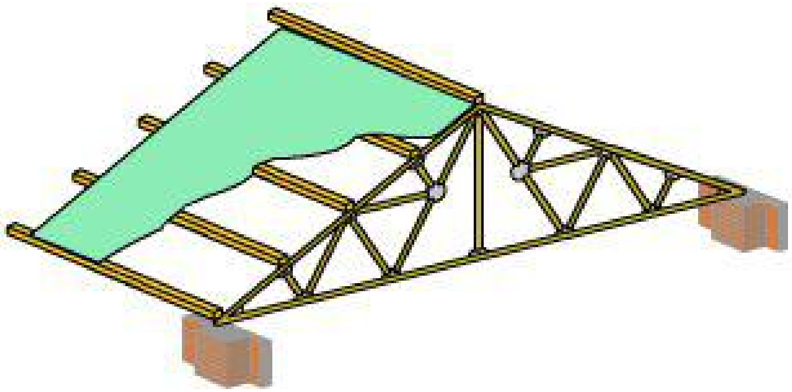
总结：

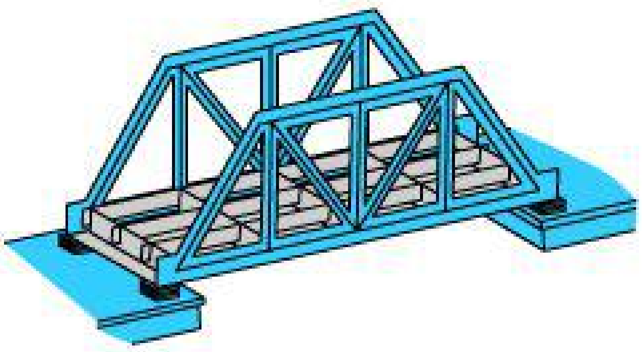
- 二力杆的分析；
- 一般不拆滑轮。

§2-6　平面简单桁架的内力计算

桁架：一种由杆件彼此在两端用铰链连接而成的结构，它在受力后几何形状不变。

节点：桁架中杆件的铰链接头。





关于平面桁架的几点假设：

1. 各杆件为直杆，各杆轴线位于同一平面内；

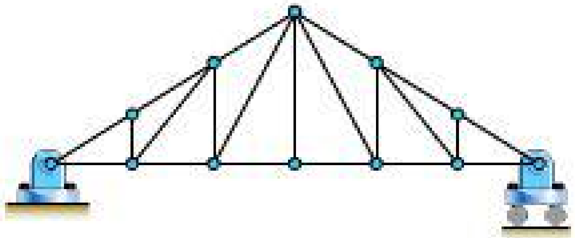
2. 杆件与杆件间均用光滑铰链连接；

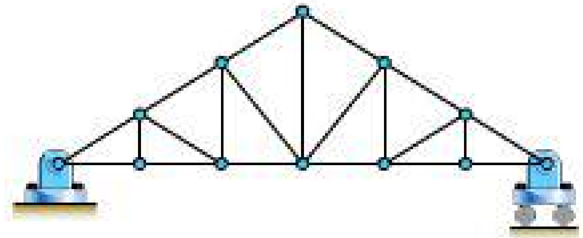
3. 载荷作用在节点上，且位于桁架几何平面内；

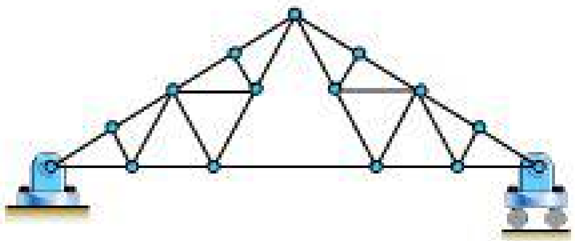
4. 各杆件自重不计或平均分布在节点上。

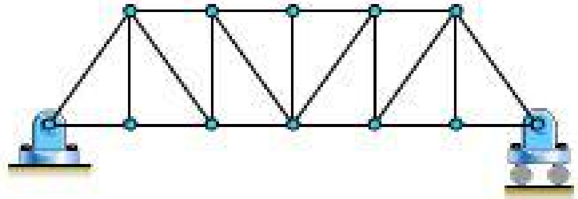
→ 理想桁架

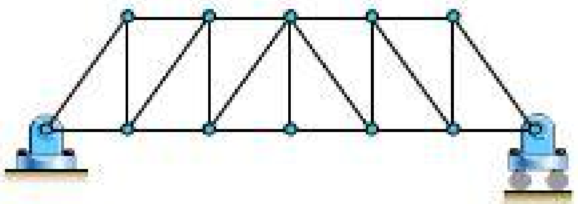
桁架中每根杆件均为二力杆

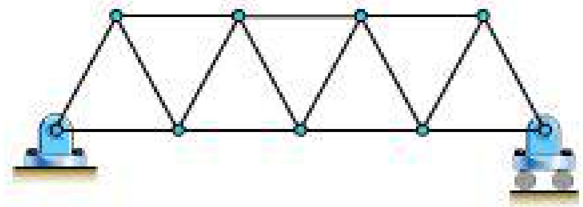


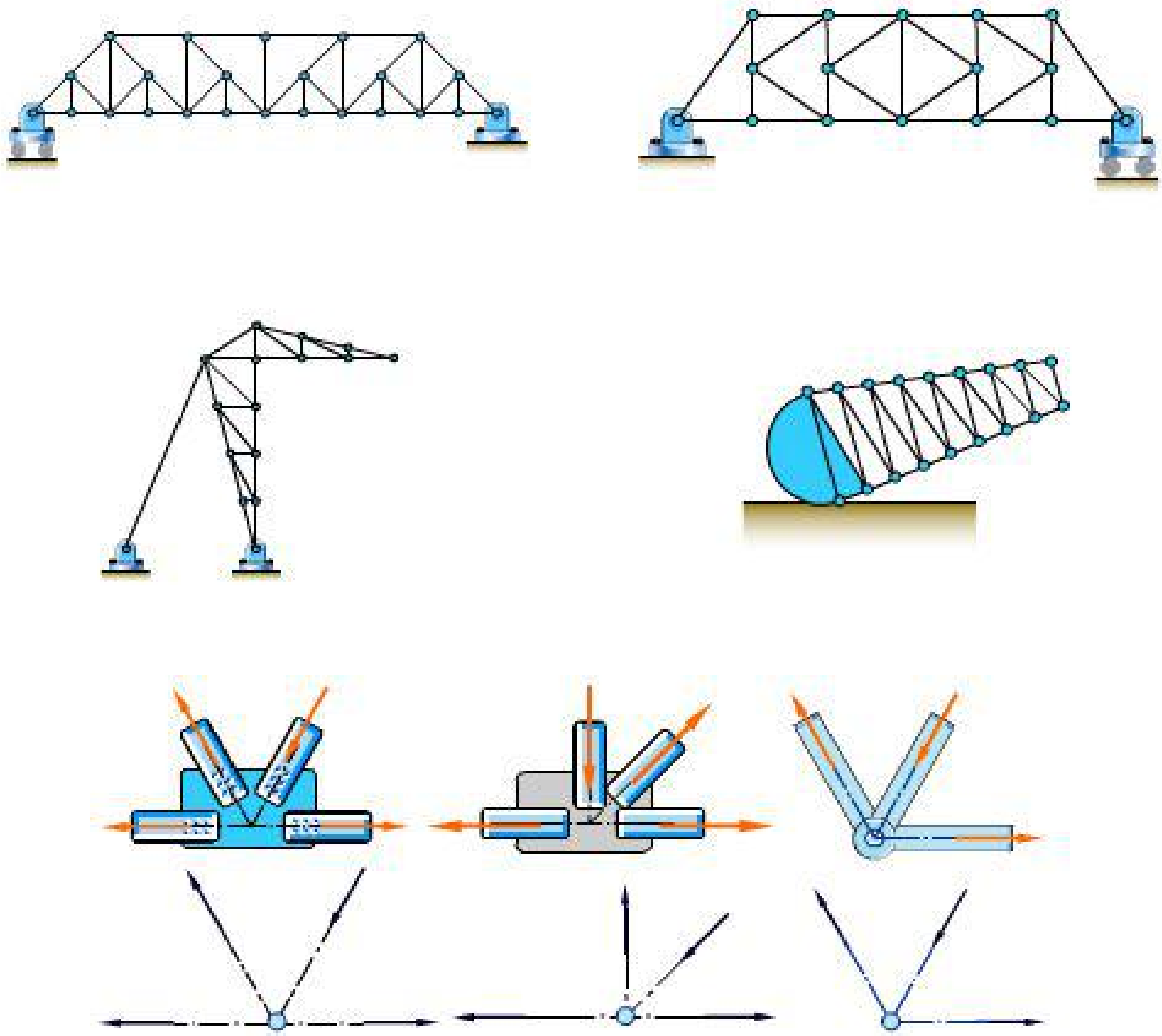




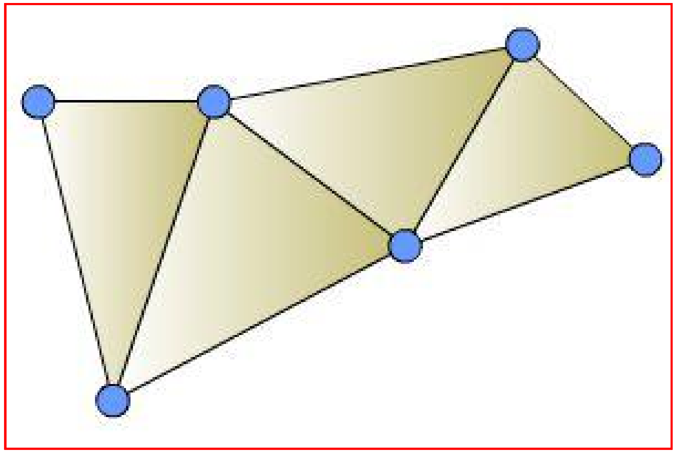






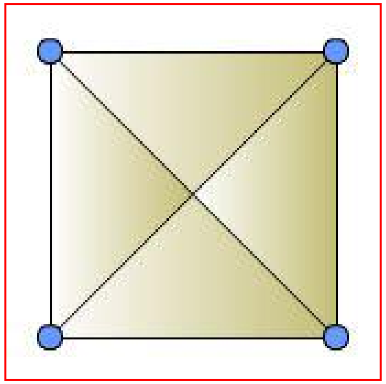


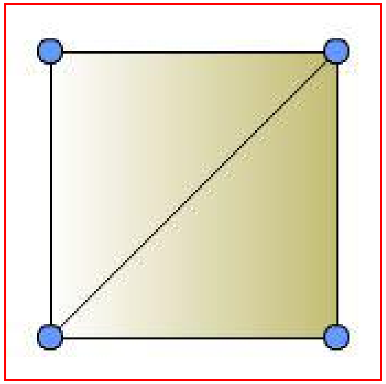
总杆数 m 总节点数 n

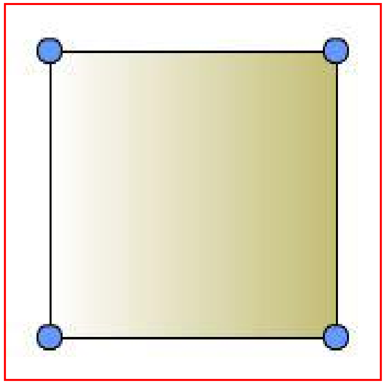


$$m-3=2(n-3)$$

$$m=2n-3$$







$m>2n-3$

平面复杂

（超静定）桁架

$m=2n-3$

平面简单

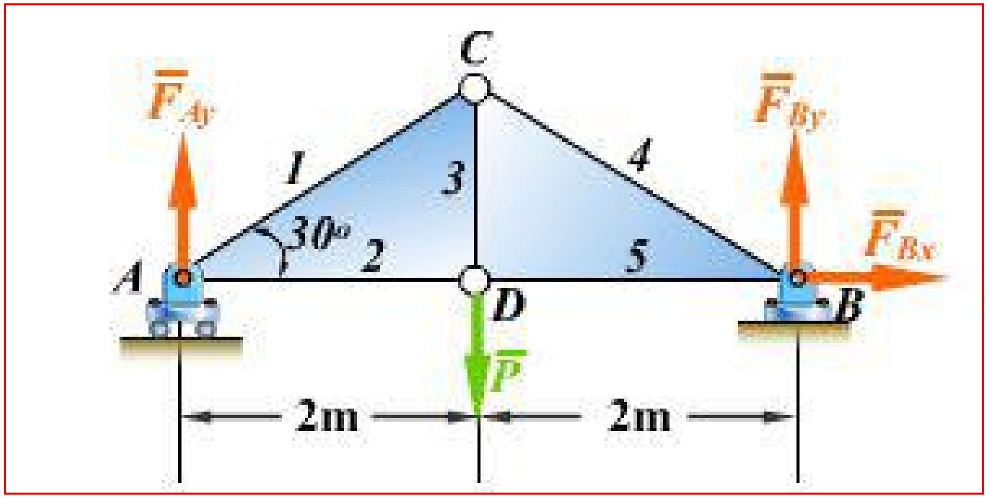
（静定）桁架

$m<2n-3$

非桁架（机构）

节点法与截面法

例2-27



已知：P=10kN, 尺寸如图；

求：　桁架各杆件受力.

解： 取整体，画受力图.

$$\sum F_x = 0 \qquad F_{Bx} = 0$$

$$\sum M_B = 0 \qquad 2P - 4F_{Ay} = 0 \qquad F_{Ay} = 5\text{kN}$$

$$\sum F_y = 0 \qquad F_{Ay} + F_{By} - P = 0 \qquad F_{By} = 5\text{kN}$$

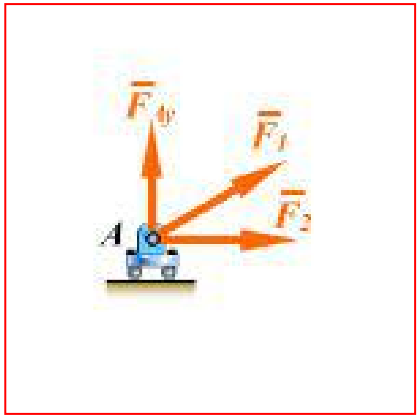
取节点A，画受力图.

节点法

$$\sum F_y = 0 \qquad F_{Ay} + F_1 \sin 30^0 = 0$$

$$\sum F_x = 0 \qquad F_2 + F_1 \cos 30^0 = 0$$

$F_1 = -10\text{kN}$（压）$F_2 = 8.66\text{kN}$（拉）

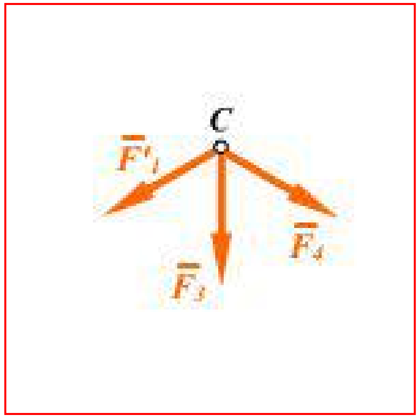


取节点C，画受力图.

$$\sum F_x = 0 \quad F_4 \cos 30^0 - F_1^{'} \cos 30^0 = 0$$

$$\sum F_y = 0 \quad -F_3 - \left(F_1^{'} + F_4\right)\sin 30^0 = 0$$

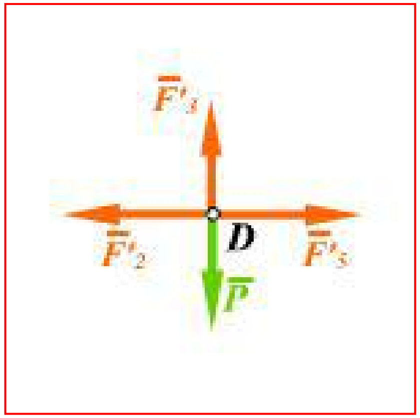
$F_4 = -10\text{kN}$（压）　$F_3 = 10\text{kN}$（拉）



取节点D，画受力图.

$$\sum F_x = 0 \quad F_5 - F_2^{'} = 0 \quad F_5 = 8.66\text{kN}$$

（拉）

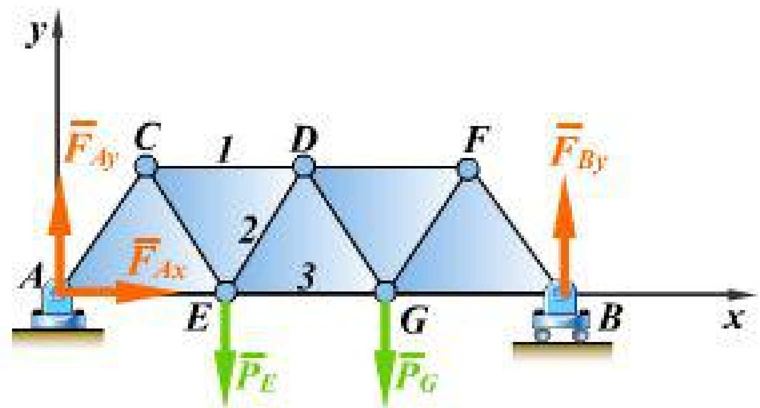


例2-28

已知：$P_E = 10\text{kN}$, $P_G = 7\text{kN}$, 各杆长度均为1m；

求：1, 2, 3杆受力.

解： 取整体, 求支座约束力.



$$\sum F_x = 0 \qquad F_{Ax} = 0$$

$$\sum M_B = 0 \quad 2P_E + P_G - 3F_{Ay} = 0$$

$$\sum F_y = 0 \quad F_{Ay} + F_{By} - P_E - P_G = 0$$

$$\Rightarrow \quad F_{Ay} = 9\text{kN} \qquad F_{By} = 8\text{kN}$$

用截面法，取桁架左边部分.

截面法

$$\sum M_E = 0 \qquad -F_1 \cdot 1 \cdot \cos 30^0 - F_{Ay} \cdot 1 = 0$$

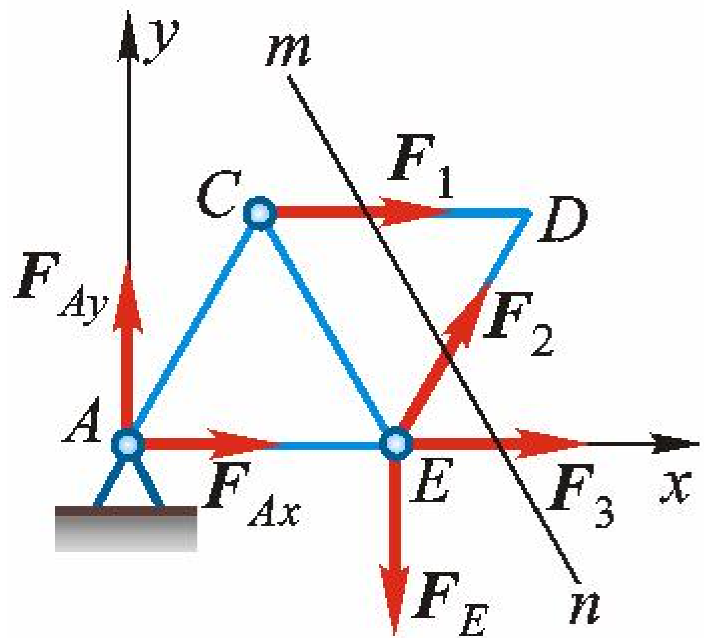
$$\sum F_y = 0 \qquad F_{Ay} + F_2 \cdot \sin 60^0 - P_E = 0$$

$$\sum F_x = 0 \qquad F_1 + F_3 + F_2 \cos 60^0 = 0$$

$F_1 = 10.4\text{kN}$（压）

$F_2 = 1.15\text{kN}$（拉）

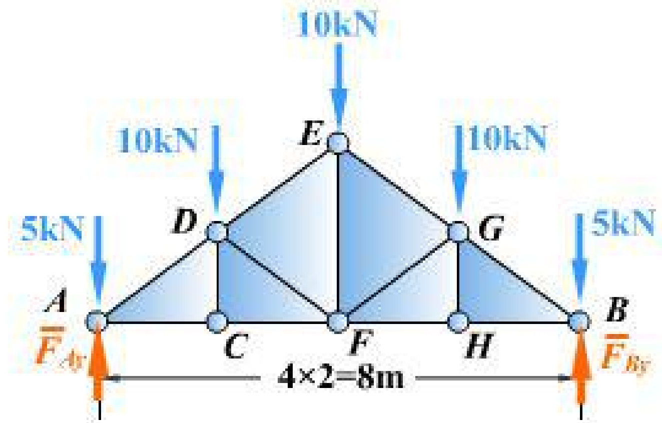
$F_3 = 9.81\text{kN}$（拉）



例2-29

已知：　荷载与尺寸如图；

求：　每根杆所受力.



解：　取整体，画受力图.

$$\sum F_x = 0 \qquad F_{Ax} = 0$$

$$\sum M_B = 0 \quad -8F_{Ay} + 5\times 8 + 10\times 6 + 10\times 4 + 10\times 2 = 0$$

得　$F_{Ay} = 20\mathbf{kN}$

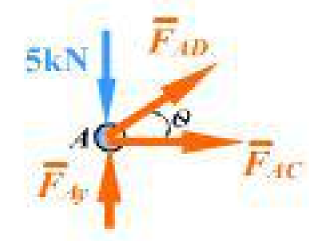
$$\sum F_y = 0 \qquad F_{Ay} + F_{By} - 40 = 0$$

得　$F_{By} = 20\text{kN}$

求各杆内力

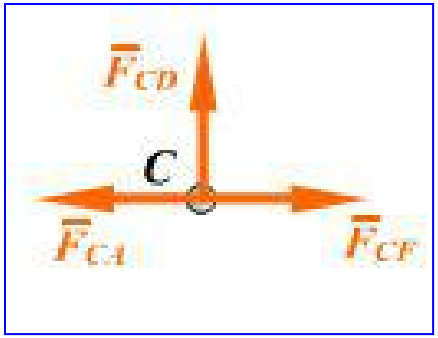
取节点*A*

$$\begin{cases} \sum F_y = 0 \rightarrow F_{AD} \\ \sum F_x = 0 \rightarrow F_{AC} \end{cases}$$



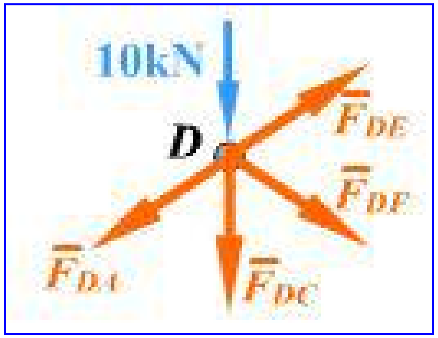
取节点C

$$\begin{cases}\sum F_x = 0 \rightarrow F_{CF} \\ \sum F_y = 0 \rightarrow F_{CD} = 0\end{cases}$$



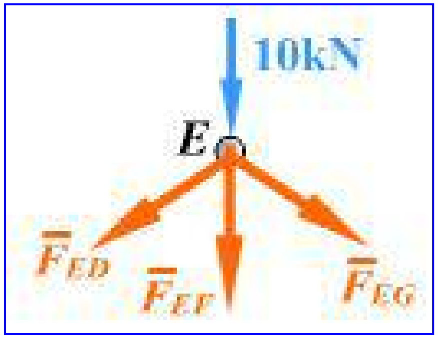
取节点D

$$\left.\begin{cases}\sum F_y = 0 \\ \sum F_x = 0\end{cases}\right. \rightarrow F_{DF}, F_{DE}$$



取节点E

$$\begin{cases}\sum F_y = 0 \rightarrow F_{EG} \\ \sum F_x = 0 \rightarrow F_{EF}\end{cases}$$

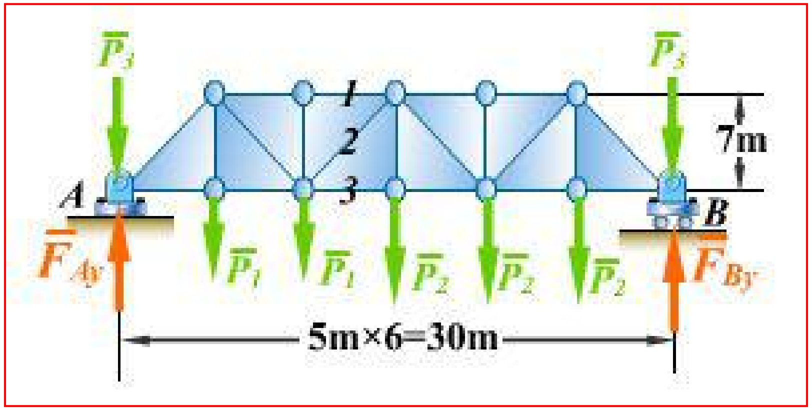


• • •   • • •

例2-30

已知：P_1, P_2, P_3，尺寸如图.

求：1， 2， 3 杆所受力.

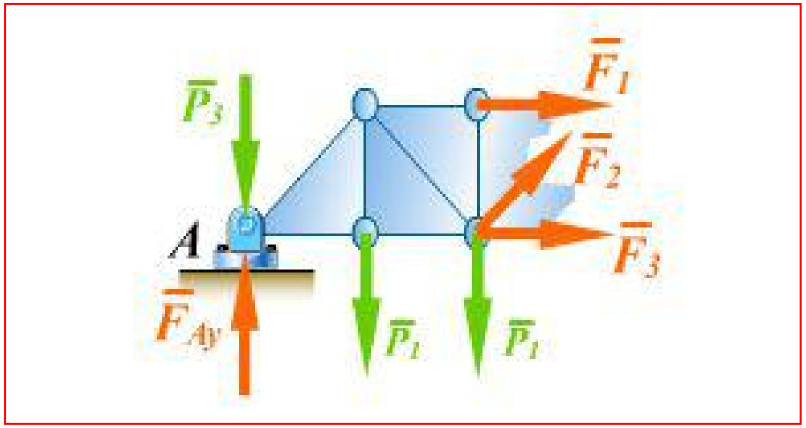


解： 求支座约束力

$$\sum M_A = 0 \rightarrow F_{Ay}$$

$$\sum F_y = 0 \rightarrow F_{By}$$

从1，2，3杆处截取左边部分

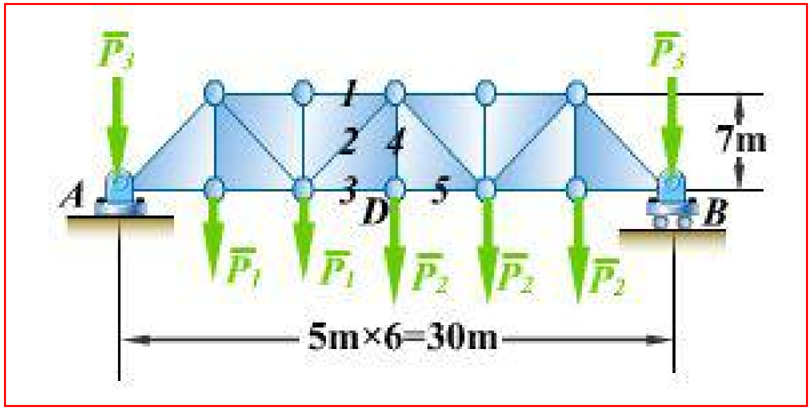


$$\sum F_y = 0 \rightarrow F_2$$

$$\sum M_C = 0 \rightarrow F_1$$

$$\sum F_x = 0 \rightarrow F_3$$

若再求 4，5 杆受力



取节点*D*

$$\sum F_x = 0 \rightarrow \quad F_5$$

$$\sum F_y = 0 \rightarrow \quad F_4$$